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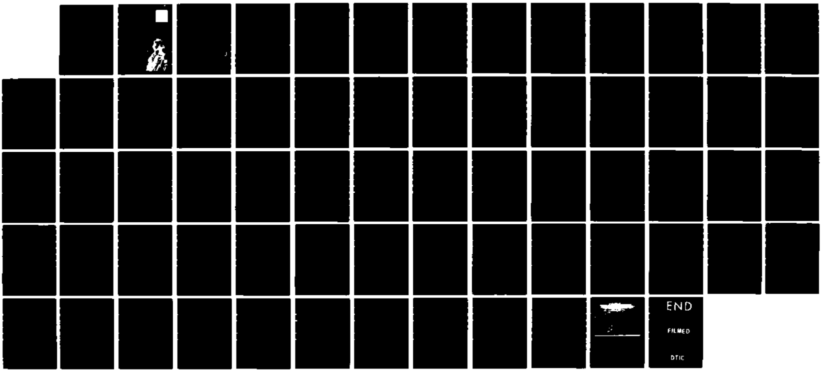

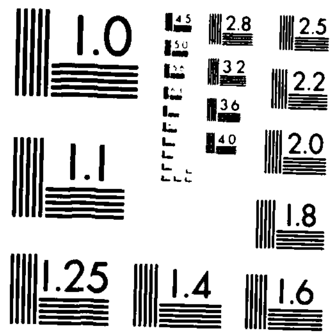


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IMETRIC MATRIX PERTURBED BY A SYMMETRIC RANK
O MATRIX COMPOSED OF TWO NONSYMMETRIC DYADS

by

Sylvia G. Leaver
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FORMULAS FOR UPDATING THE GENERALIZED INVERSE OF A SYMMETRIC MATRIX PERTURBED BY A SYMMETRIC RANK TWO MATRIX COMPOSED OF TWO NONSYMMETRIC DYADS

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GWU/IMSE/Serial-T-493/84
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School of Engineering and Applied Science
Institute for Management Science and Engineering



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THE GEORGE WASHINGTON UNIVERSITY
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Abstract
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# FORMULAS FOR UPDATING THE GENERALIZED INVERSE OF A SYMMETRIC MATRIX PERTURBED BY A SUMMETRIC RANK TWO MATRIX COMPOSED OF TWO NONSYMMETRIC DYADS

by

Sylvia G. Leaver
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In many applications employing a symmetric matrix and its generalized (Penrose-Moore) inverse the matrix is given in a natural way as the finite sum of symmetric dyadic matrices and pairs of nonsymmetric dyadic matrices. In this paper, formulas are given for the generalized inverse, $B^{+}$, of $B = A + ab^T + ba^T$ for $A$ symmetric, $a,b$ vectors. There are nine distinct cases which must be considered. The application of these formulas is given to the computation of an estimate of the positive part and directions of negative curvature for a symmetric matrix.

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# FORMULAS FOR UPDATING THE GENERALIZED INVERSE OF A SYMMETRIC MATRIX PERTURBED BY A SYMMETRIC RANK TWO MATRIX COMPOSED OF TWO NONSYMMETRIC DYADS*

by

Sylvia G. Leaver
Garth P. McCormick

## 1. Introduction

In many applications employing a symmetric matrix and its generalized inverse, the matrix is given in a natural way as the finite sum of symmetric dyadic matrices and pairs of nonsymmetric dyadic matrices. That is, if A is a symmetric matrix, then A is given as:

$$A = \sum_{i=1}^{p} d_i c_i d_i^T + \sum_{j=1}^{q} a_j b_j^T + b_j a_j^T \tag{1}$$

for some $p$ and $q$ integers greater than or equal to 0, where each $d_i$, $a_j$ and $b_j$ are $n \times 1$ vectors and each $c_i$ is a scalar. For example, the Hessian matrix of the product of two linear functions $f(x)$ and $g(x)$ :

$$\begin{aligned} \nabla^2[f(x)g(x)] &= g(x)\nabla^2 f(x) + f(x)\nabla^2 g(x) \\ &\qquad + \nabla g(x)\big(\nabla f(x)\big)^T + \nabla f(x)\big(\nabla g(x)\big)^T \\ &= \nabla g(x)\big(\nabla f(x)\big)^T + \nabla f(x)\big(\nabla g(x)\big)^T \end{aligned}$$

has this structure.

*Much of this material is taken from Leaver (1984).

The generalized (Penrose-Moore) inverse of a matrix A is the unique matrix, $A^+$ , which satisfies the following four properties:

$$AA^+A = A \tag{2}$$

$$A^+AA^+ = A^+ \tag{3}$$

$$(A^+A)^T = A^+A \tag{4}$$

$$(AA^+)^T = AA^+ . \tag{5}$$

Ben-Israel and Greville (1974) provide existence and uniqueness proofs for the Penrose-Moore inverse.

When A is expressed as in (1), $A^+$ can be computed recursively. In particular, any pair of nonsymmetric dyads can be written as:

$$ab^T + ba^T = u \frac{1}{2k} u^T - v \frac{1}{2k} v^T ,$$

where

$$u = a + bk ,$$

$$v = a - bk ,$$

$$k \neq 0 ,$$

and $A^+$ can be computed recursively as the generalized inverse of a matrix A perturbed by a series of symmetric dyads. Formulas for updating $A^+$ when A is perturbed by a single dyad were found by Meyer (1973) and McCormick (1976). Meyer's results cover the general case when neither A nor the dyadic perturbation is square and symmetric. McCormick's formulas are special cases of Meyer's results and apply when both A and the dyadic perturbation are symmetric.

In the following, formulas are given for the generalized inverse, $B^+$ , of $B = A + ab^T + ba^T$ for A symmetric. There are nine distinct cases which must be considered.

## 2. Formulas for Updating a Generalized Inverse

We begin by first defining a matrix $P$ which maps every vector into the null space of $A$ as:

$$P = I - A^+A .$$

The following relations hold:

(i) $P^T = I - (A^+A)^T = I - A^+A = P$ by (5). (6)

(ii) $AP = A - AA^+A = 0$ , by (2). (7)

(iii) $PA = (AP)^T = 0$ , by (7) and the symmetry of $A$ and $P$ . (8)

(iv) $(A^+)^T = A^+$ . (9)

Proof:

$(AA^+A)^T = AA^{+T}A = A$ by (2).

$(A^+AA^+)^T = (A^+)^T A(A^+)^T = A^{+T}$ by (3).

$\left((A^+)^T A\right)^T = A^T A^+ = AA^+ = (AA^+)^T$ by (4),

$= (A^+)^T A$ .

$\left(A(A^+)^T\right)^T = A^+A^T = A^+A = (A^+A)^T$ by (5),

$= A(A^+)^T$ .

Hence $(A^+)^T$ is a Penrose-Moore inverse of $A$ and $(A^+)^T = A^+$ by the uniqueness property of $A^+$ .

(v) $(AA^+) = (AA^+)^T = (A^+)^T A^T = A^+A$ , by (4), (9) and the symmetry of $A$ . Thus (10)

(vi) $P = I - AA^+$ , by (10). (11)

(vii) $A^+P = A^+ - A^+AA^+ = 0 = (A^+P)^T = PA^+$ . (12)

It also will be useful to recall from Halmos (1958),

(viii) If $A = A^T$ , $B = B^T$ , then

$$AB = (AB)^T \ \left(\text{and} \ \ BA = (BA)^T\right) \Leftrightarrow AB = BA \ . \tag{13}$$

(ix) $\text{Rank}(A+B) \leqslant \text{Rank}(A) + \text{Rank}(B)$ , and (14)

$\text{Rank}(A+B) \geqslant |\text{Rank}(A) - \text{Rank}(B)|$ .

(x) For $A$ an $n \times n$ matrix, (15)

$\text{Rank}(A) + \text{Nullity}(A) = n$ .

<u>Theorem 1</u>:

If

$$Pa = 0 \tag{16}$$

$$Pb = 0 \tag{17}$$

and

$$a^TA^+a \ b^TA^+b - (a^TA^+b+1)^2 \neq 0 \ , \tag{18}$$

then

$$\text{Rank}(B) = \text{Rank}(A) \ , \tag{19}$$

and

$$B^+ = (A+ab^T+ba^T)^+ = \left[A + (a,b)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{pmatrix}a^T\\ b^T\end{pmatrix}\right]^+$$

$$= A^+ - A^+(a,b)\left[\begin{pmatrix}a^T\\ b^T\end{pmatrix}A^+(a,b) + \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right]^{-1}\begin{pmatrix}a^T\\ b^T\end{pmatrix}A^+ = G \ . \tag{20}$$

Proof: Note first that by (16),

$$\left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right]$$

has an inverse, and from (16) and (17) it follows that

$$AA^+a = A^+A = a \ , \tag{21}$$

and

$$AA^+b = A^+Ab = b \ . \tag{22}$$

Now

$$BG = \left[A + (a,b)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} a^T \\ b^T \end{pmatrix}\right] \cdot$$

$$\left[A^+ - A^+(a,b)\left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right]^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+\right]$$

$$= AA^+ + AA^+(a,b)\left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right]^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+$$

$$+ (a,b)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+$$

$$- (a,b)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b)\right]\left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right]^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+$$

$$= AA^+ + (a,b)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+$$

$$- (a,b)\left[I + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b)\right]\left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right]^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ \ .$$

Now

$$\left[ I + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b) \right] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left[ \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right]$$

so

$$BG = AA^+ + (ab) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(I-I) = AA^+ . \tag{23}$$

Hence $(BG)^T = BG$ by (4). Let $Q = (I-BG)$ . Now by (23), $Q = P$ , so by (15) it follows that

$$\text{Rank}(B) = \text{Rank}(A) .$$

Then

$$\begin{aligned} BGB &= (AA^+)(A + ab^T + ba^T) \\ &= AA^+A + AA^+ab^T + AA^+ba^T \\ &= A + ab^T + ba^T \equiv B \end{aligned}$$

by (2), (21), and (22). Thus (2) holds. Noting that $G$ is symmetric by (20), we then have by (13) that

$$\begin{aligned} BG = (BG)^T &\Leftrightarrow GB = BG \\ &\Rightarrow (GB)^T = GB , \end{aligned}$$

and (5) holds. Then

$$\begin{aligned} GBG &= \left[ A^+ - A^+(a,b) \left[ \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right]^{-1} \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ \right] \cdot AA^+ \\ &= A^+AA^+ - A^+(a,b) \left[ \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right]^{-1} \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+AA^+ \\ &= A^+ - A^+(a,b) \left[ \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right]^{-1} \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ \quad \text{by (3)} \\ &\equiv G , \text{ and (3) holds.} \end{aligned}$$

Theorem 2:

If

$$Pa = 0 \ , \tag{24}$$

$$Pb = 0 \ , \tag{25}$$

$$a^T A^+ a b^T A^+ b - (a^T A^+ b + 1)^2 = 0 \ , \tag{26}$$

$$a^T A^+ a = 0 \ , \tag{27}$$

and

$$b^T A^+ b = 0 \ , \tag{28}$$

then

$$\mathrm{Rank}(B) = \mathrm{Rank}(A) - 2 \ , \tag{29}$$

and

$$\begin{aligned} B^+ = A^+ &- A^+A^+(a,b)\left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+A^+(a,b)\right]^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ \\ &- A^+(a,b)\left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+A^+(a,b)\right]^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+A^+ \\ &+ A^+(a,b)\left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+A^+(a,b)\right]^{-1}\left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+A^+A^+(a,b)\right] \\ &\cdot \left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+A^+(a,b)\right]^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ \\ = G \ . \end{aligned} \tag{30}$$

Proof: Note as in Theorem 1, (24) and (25) imply that

$$A^+Aa = AA^+a = a \ , \tag{31}$$

and

$$A^+Ab = AA^+b = b \ , \tag{32}$$

and from (26), (27) and (28) it follows that

$$a^T A^+ b = -1 \; . \tag{33}$$

Let

$$F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; , \quad C = \left[ \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ A^+ (a,b) \right] \; , \quad \text{and} \quad D = \left[ \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ A^+ A^+ (a,b) \right] \; .$$

Note also that $a^T A^+ A^+ a \neq 0$ (and similarly $b^T A^+ A^+ b \neq 0$), since $A^+ a = 0 \Rightarrow Pa = a \neq 0$ , a contradiction, and that

$$a^T A^+ \neq k b^T A^+ \qquad \text{for any } k \neq 0 \; , \tag{34}$$

since

$$a^T A^+ = k b^T A^+ \Rightarrow a^T A^+ a = k b^T A^+ a \Rightarrow 0 = k \,-1 \Rightarrow k = 0 \; ,$$

another contradiction. Hence

$$\det C = a^T A^+ A^+ a \; b^T A^+ A^+ b - (a^T A^+ A^+ b)^2 \neq 0$$

and $C$ has an inverse. Now

$$\begin{aligned} BG = & \left[ A + (a,b) F \begin{pmatrix} a^T \\ b^T \end{pmatrix} \right] \cdot \\ & \left[ A^+ - A^+ A^+ (a,b) C^{-1} \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ - A^+ (a,b) C^{-1} \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ A^+ \right. \\ & \left. + A^+ (a,b) C^{-1} D C^{-1} \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ \right] \\ = & AA^+ - AA^+ A^+ (a,b) C^{-1} \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ - AA^+ (a,b) C^{-1} \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ A^+ \\ & + AA^+ (a,b) C^{-1} D C^{-1} \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ + (a,b) F \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+ \end{aligned}$$

$$- (a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+A^+(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+$$

$$- (a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+A^+$$

$$+ (a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+(a,b)C^{-1}DC^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+$$

$$= AA^+ - A^+(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+ - (a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+A^+$$

$$+ (a,b)C^{-1}DC^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+ + (a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+$$

$$- (a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+ - (a,b)F\left[\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+(a,b)\right]C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+A^+$$

$$+ (a,b)F\left[\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+(a,b)\right]C^{-1}DC^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+ ,$$

by (2), (10), (30) and (31),

$$= AA^+ - A^+(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+$$

$$- (a,b)\left[I + F\left[\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+(a,b)\right]\right]C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+A^+$$

$$+ (a,b)\left[I + F\left[\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+(a,b)\right]\right]C^{-1}DC^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+ .$$

Now

$$I + F\left[\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+(a,b)\right] = I + \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix} \quad \text{by (29)},$$

$$= I - I = 0 . \qquad (35)$$

$$- \left(A^+a - Pb(\alpha + a^TA^+b + 1)\cdot\zeta^{-1}\right)\cdot\alpha^{-1}\left(a^TA^+ - \zeta^{-1}(\alpha + a^TA^+b + 1)b^TP\right)$$

$$= G \ , \qquad (66)$$

where

$$\alpha = a^TA^+a$$

$$\zeta = b^TPb \ .$$

Proof: Note that by (63) and (64), $b^TPb = b^TPPb > 0$ , and $a^TA^+a \neq 0$ and thus have inverses. Now

$$\begin{aligned}
BG = {} & (A + ab^T + ba^T)\Big(A^+ - A^+(a+b)\zeta^{-1}b^TP - Pb\zeta^{-1}(a+b)^TA^+ \\
& + Pb\left((a+b)^TA^+(a+b) + 2\right)\zeta^{-2}b^TP \\
& - \left(A^+a - Pb(\alpha + a^TA^+b + 1)\zeta^{-1}\right)\alpha^{-1}\left(a^TA^+ - \zeta^{-1}(\alpha + a^TA^+b + 1)b^TP\right)\Big) \\
= {} & AA^+ - AA^+(a+b)\zeta^{-1}b^TP - (AA^+a)\alpha^{-1}\left(a^TA^+ - \zeta^{-1}(\alpha + a^TA^+b + 1)b^TP\right) \\
& + ab^TA^+ - ab^TA^+(a+b)\zeta^{-1}b^TP - ab^TPb\zeta^{-1}(a+b)^TA^+ \\
& + ab^TPb\left((a+b)^TA^+(a+b) + 2\right)\zeta^{-2}b^TP \\
& - a\left(b^TA^+a - b^TPb(\alpha + a^TA^+b + 1)\zeta^{-1}\right)\alpha^{-1}\left(a^TA^+ - \zeta^{-1}(\alpha + a^TA^+b + 1)b^TP\right) \\
& + ba^TA^+ - ba^TA^+(a+b)\zeta^{-1}b^TP \\
& - b(a^TA^+a)\alpha^{-1}\left(a^TA^+ - \zeta^{-1}(\alpha + a^TA^+b + 1)b^TP\right) \\
= {} & AA^+ - a\zeta^{-1}b^TP - AA^+b\zeta^{-1}b^TP \\
& - a\alpha^{-1}a^TA^+ + a\alpha^{-1}\zeta^{-1}(\alpha + a^TA^+b + 1)b^TP + ab^TA^+ \\
& - a(b^TA^+a + b^TA^+b)\zeta^{-1}b^TP - aa^TA^+ - ab^TA^+ \\
& + a\left((a+b)^TA^+(a+b) + 2\right)\zeta^{-1}b^TP \\
& + a(\alpha+1)\alpha^{-1}a^TA^+
\end{aligned}$$

Note also that

$$B(A^+v) = 0 .$$

Proof:

$$\left(A + u\frac{1}{2k}u^T - v\frac{1}{2k}v^T\right)A^+v = AA^+v + u\frac{1}{2k}u^TA^+v - v\frac{1}{2k}v^TA^+v$$

$$= v - v = 0$$

by (46), (47), (57) and (58). But also by (46) and (47),

$$AA^+v = v \neq 0 .$$

So, as in Theorem 3,

$$\text{Rank}(Q) = \text{Rank}(P) + 1$$

and by (15) we have

$$\text{Rank}(B) = \text{Rank}(A) - 1 .$$

Theorem 5:

If

$$Pa = 0 , \tag{62}$$

$$Pb \neq 0 , \tag{63}$$

and

$$a^TA^+a \neq 0 , \tag{64}$$

then

$$\text{Rank}(B) = \text{Rank}(A) + 1 , \tag{65}$$

and

$$B^+ = A^+ - A^+(a+b)\zeta^{-1}b^TP - Pb\zeta^{-1}(a+b)^TA^+$$

$$+ Pb\left((a+b)^TA^+(a+b) + 2\right)\zeta^{-2}b^TP$$

$$- A^+\big(u - v\delta^{-1}(v^TA^+A^+u)\big)(4a^TA^+a)^{-1}\big(u - v\delta^{-1}(v^TA^+A^+u)\big)^TA^+\big)$$

$$= A^+AA^+ - A^+AA^+A^+v\delta^{-1}v^TA^+ - A^+AA^+v\delta^{-1}v^TA^+A^+$$

$$+ A^+AA^+v\delta^{-2}(v^TA^+A^+A^+v)v^TA^+$$

$$- A^+AA^+\big(u - v\delta^{-1}(v^TA^+A^+u)\big)(4a^TA^+a)^{-1}\big(u - v\delta^{-1}(v^TA^+A^+u)\big)^TA^+$$

$$- A^+v\delta^{-1}v^TA^+A^+ + A^+v\delta^{-2}(v^TA^+A^+A^+v)v^TA^+$$

$$+ A^+v\delta^{-2}(v^TA^+A^+v)v^TA^+A^+ - A^+v\delta^{-3}(v^TA^+A^+v)(v^TA^+A^+A^+v)v^TA^+$$

$$+ A^+v\delta^{-1}v^TA^+A^+\big(u - v\delta^{-1}(v^TA^+A^+u)\big)(4a^TA^+a)^{-1}\big(u - v\delta^{-1}(v^TA^+A^+u)\big)^TA^+$$

$$= A^+ - A^+A^+v\delta^{-1}v^TA^+ - A^+v\delta^{-1}v^TA^+A^+$$

$$+ A^+v\delta^{-2}(v^TA^+A^+A^+v)v^TA^+$$

$$- A^+\big(u - v\delta^{-1}(v^TA^+A^+u)\big)(4a^TA^+a)^{-1}\big(u - v\delta^{-1}(v^TA^+A^+u)\big)^TA^+$$

$$+ A^+v\delta^{-1}\big(v^TA^+A^+u(1-1)\big)(4a^TA^+a)^{-1}\big(u - v\delta^{-1}(v^TA^+A^+u)\big)^TA^+$$

$$= A^+ - A^+A^+v\delta^{-1}v^TA^+ - A^+v\delta^{-1}v^TA^+A^+$$

$$+ A^+v\delta^{-2}(v^TA^+A^+A^+v)v^TA^+$$

$$- A^+\big(u - v\delta^{-1}(v^TA^+A^+u)\big)(4a^TA^+a)^{-1}\big(u - v\delta^{-1}(v^TA^+A^+u)\big)^TA^+$$

$$\equiv G \,.$$

Let $Q = I - BG$ . Then by (61),

$$Q = P + A^+v\delta^{-1}v^TA^+ \,.$$

Now (46) and (47) imply that

$$Bx = 0 \quad \text{for all } x \text{ satisfying } Ax = 0 \,,$$

which implies

$$\mathrm{Rank}(Q) \geqslant \mathrm{Rank}(P) \,.$$

$$1 - \frac{1}{2k} 4a^T A^+ a + \frac{1}{2k} u^T A^+ u = 1 - \frac{1}{2k} (4a^T A^+ a - 4a^T A^+ a + 2k)$$

$$= 1 - \frac{1}{2k} (2k) = 0 \ . \tag{60}$$

So (60) gives us

$$BG = AA^+ - A^+ v\delta^{-1} v^T A^+ \ , \tag{61}$$

which is summetric by (4). Note

$$G = G^T \quad \text{and} \quad B = B^T$$

give us by (13) and (45) that

$$GB = BG = (GB)^T$$

and (5) holds. Now

$$BGB = (AA^+ - A^+ v\delta^{-1} v^T A^+)(A - v \frac{1}{2k} v^T + u \frac{1}{2k} u^T)$$

$$= AA^+ A - AA^+ v \frac{1}{2k} v^T + AA^+ u \frac{1}{2k} u^T$$

$$- A^+ v\delta^{-1} v^T A^+ A + A^+ v\delta^{-1} v^T A^+ v \frac{1}{2k} v^T$$

$$- A^+ v\delta^{-1} v^T A^+ u \frac{1}{2k} u^T \ ,$$

which by (46), (47), and (57)

$$= A - v \frac{1}{2k} v^T + u \frac{1}{2k} u^T$$

$$- A^+ v\delta^{-1} v^T + A^+ v\delta^{-1} v^T$$

$$= A - v \frac{1}{2k} v^T + u \frac{1}{2k} u^T \equiv B \ .$$

And

$$GBG = (A^+ A - A^+ v\delta^{-1} v^T A^+)\big(A^+ - A^+ A^+ v\delta^{-1} v^T A^+ - A^+ v\delta^{-1} v^T A^+ A^+$$

$$+ A^+ v\delta^{-2} (v^T A^+ A^+ A^+ v) v^T A^+$$

$$a^TA^+a = k^2b^TA^+b \ , \ (a^TA^+b + 1) = kb^TA^+b$$

$$\Rightarrow u^TA^+v = a^TA^+a - k^2b^TA^+b = 0 \ , \tag{57}$$

$$v^TA^+v = a^TA^+a + k^2b^TA^+b - 2ka^TA^+b = 2k \ , \text{ and} \tag{58}$$

$$u^TA^+u = a^TA^+a + k^2b^TA^+b + 2ka^TA^+b = 4a^TA^+a - 2k \tag{59}$$

so

$$\begin{aligned}
BG = AA^+ &- A^+v\delta^{-1}v^TA^+ - v\delta^{-1}v^TA^+A^+ \\
&+ v\delta^{-2}(v^TA^+A^+A^+v)v^TA^+ \\
&- v\,\frac{1}{2k}\,v^TA^+ + v\,\frac{1}{2k}\,v^TA^+A^+v\delta^{-1}v^TA^+ \\
&+ v\,\frac{1}{2k}\,v^TA^+v\delta^{-1}v^TA^+A^+ - v\,\frac{1}{2k}\,v^TA^+v\delta^{-2}(v^TA^+A^+A^+v)v^TA^+ \\
&- \big(u - v\delta^{-1}(v^TA^+A^+u)\big)(4a^TA^+a)^{-1}\big(u - v\delta^{-1}(v^TA^+A^+u)\big)^TA^+ \\
&+ v\,\frac{1}{2k}\,v^TA^+\big(u - v\delta^{-1}(v^TA^+A^+u)\big)(4a^TA^+a)^{-1}\big(u - v\delta^{-1}(v^TA^+A^+u)\big)^TA^+ \\
&+ u\,\frac{1}{2k}\,u^TA^+ - u\,\frac{1}{2k}\,u^TA^+A^+v\delta^{-1}v^TA^+ \\
&- u\,\frac{1}{2k}\,u^TA^+u(4a^TA^+a)^{-1}\big(u - v\delta^{-1}(v^TA^+A^+u)\big)^TA^+ \\
= AA^+ &- A^+v\delta^{-1}v^TA^+ \\
&+ u\big(-(4a^TA^+a)^{-1}(1 - \frac{1}{2k}\,4a^TA^+a + \frac{1}{2k}\,u^TA^+u)\big)u^TA^+ \\
&+ u\big(\delta^{-1}(v^TA^+A^+u)(4a^TA^+a)^{-1}(1 - \frac{1}{2k}\,4a^TA^+a + \frac{1}{2k}\,u^TA^+u)\big)v^TA^+ \\
&+ v\big(\delta^{-1}(v^TA^+A^+u)(4a^TA^+a)^{-1}(1 - 1)\big)u^TA^+ \\
&+ v\big(\delta^{-2}(v^TA^+A^+u)^2(4a^TA^+a)^{-1}(-1 + 1)\big)v^TA^+.
\end{aligned}$$

Now

$$a^TA^+ab^TA^+b = (a^TA^+b + 1)^2 \Rightarrow a^TA^+a = k^2b^TA^+b \tag{54}$$

and also that

$$a^TA^+b + 1 = kb^TA^+b \Rightarrow k = (a^TA^+b + 1)/b^TA^+b \ . \tag{55}$$

Note also that

$$B = A + ab^T + ba^T = A + u\,\frac{1}{2k}\,u^T - v\,\frac{1}{2k}\,v^T \ . \tag{56}$$

So we have

$$\begin{aligned}
BG = {} & \left[A - v\,\frac{1}{2k}\,v^T + u\,\frac{1}{2k}\,u^T\right]\Big[A^+ - A^+A^+v\delta^{-1}v^TA^+ - A^+v\delta^{-1}v^TA^+A^+ \\
& + A^+v\delta^{-2}(v^TA^+A^+A^+v)v^TA^+ \\
& - A^+\big(u - v\delta^{-1}(v^TA^+A^+u)\big)(4a^TA^+a)^{-1}\big(u - v\delta^{-1}(v^TA^+A^+u)\big)^TA^+\Big] \\
= {} & \left[A - v\,\frac{1}{2k}\,v^T\right]\big(A^+ - A^+A^+v\delta^{-1}v^TA^+ - A^+v\delta^{-1}v^TA^+A^+ \\
& + A^+v\delta^{-2}(v^TA^+A^+A^+v)v^TA^+\big) \\
& - \left[A - v\,\frac{1}{2k}\,v^T\right]A^+\big(u - v\delta^{-1}(v^TA^+A^+u)\big)(4a\ A\ a)^{-1} \\
& \quad \cdot \big(u - v\delta^{-1}(v^TA^+A^+u)\big)^TA^+ \\
& + u\,\frac{1}{2k}\,u^TA^+ - u\,\frac{1}{2k}\,u^TA^+A^+v\delta^{-1}v^TA^+ \\
& - u\,\frac{1}{2k}\,u^TA^+v\delta^{-1}v^TA^+A^+ \\
& + u\,\frac{1}{2k}\,u^TA^+v\delta^{-2}(v^TA^+A^+A^+v)v^TA^+ \\
& - u\,\frac{1}{2k}\,u^TA^+\big(u - v\delta^{-1}(v^TA^+A^+u)\big)(4a^TA^+a)^{-1} \\
& \quad \cdot \big(u - v\delta^{-1}(v^TA^+A^+u)\big)^TA^+ \ .
\end{aligned}$$

Now

Theorem 4:

If

$$Pa = 0 \ , \tag{46}$$

$$Pb = 0 \ , \tag{47}$$

$$a^T A^+ a b^T A^+ b - (a^T A^+ b + 1)^2 = 0 \ , \tag{48}$$

$$a^T A^+ a \neq 0 \ , \tag{49}$$

$$b^T A^+ b \neq 0 \ , \tag{50}$$

then

$$\mathrm{Rank}(B) = \mathrm{Rank}(A) - 1 \ , \tag{51}$$

and

$$\begin{aligned} B^+ &= A^+ - A^+A^+v\delta^{-1}v^TA^+ - A^+v\delta^{-1}v^TA^+A^+ \\ &\quad + A^+v\delta^{-2}(v^TA^+A^+A^+v)v^TA^+ \\ &\quad - A^+\left(u - v\delta^{-1}(v^TA^+A^+u)\right)(4a^TA^+a)^{-1}\left(u - v\delta^{-1}(v^TA^+A^+u)\right)^TA^+ \\ &= G \ , \end{aligned} \tag{52}$$

where

$$u = a + bk \ ,$$

$$v = a - bk \ ,$$

$$k = (a^TA^+b + 1) / b^TA^+b \ , \text{ and}$$

$$\delta = v^TA^+A^+v \ .$$

Proof: Note first that (48), (49) and (50) imply that

$$a^TA^+b \neq -1 \ , \tag{53}$$

and that

$$- A^+\big(a(a^TA^{+T}A^+b)\beta^{-1} - b\big)\gamma^{-1}\big(a(a^TA^{+T}A^+b)\beta^{-1} - b\big)^TA^{+T}$$

$$+ A^+a\beta^{-1}(1-1)a^TA^{+T}A^+ + A^+a\beta^{-2}(a^TA^{+T}A^+A^+a)(1-1)a^TA^{+T}$$

$$+ A^+a\beta^{-1}\big((a^TA^{+T}A^+a)(a^TA^{+T}A^+b)\beta^{-1} - (a^TA^{+T}A^+b)\big)\gamma^{-1}$$

$$\cdot \big(a(a^TA^{+T}A^+b)\beta^{-1} - b\big)^TA^{+T},$$

by (3),

$$= A^+ - A^+A^+a\beta^{-1}a^TA^{+T} - A^+a\beta^{-1}a^TA^{+T}A^+ + A^+a\beta^{-2}(a^TA^{+T}A^+A^+a)a^TA^{+T}$$

$$- A^+\big(a(a^TA^{+T}A^+b)\beta^{-1} - b\big)\gamma^{-1}\big(a(a^TA^{+T}A^+b)\beta^{-1} - b\big)^TA^{+T}$$

$$\equiv G \ .$$

Let $Q = I - BG$ , which by (45)

$$= P + A^+a\beta^{-1}a^TA^+ \ .$$

Now from (37) and (38) it follows that

$$Bx = 0 \quad \text{for all} \quad x \quad \text{satisfying} \quad Ax = 0 \ ,$$

so

$$\text{Rank}(Q) \geqslant \text{Rank}(P) \ .$$

Now by the same argument as in Theorem 2,

$$B(A^+a) = 0 \ ,$$

but

$$AA^+a = a \neq 0$$

by (37). So it follows that

$$\text{Rank}(Q) \geqslant \text{Rank}(P) + 1 \ ,$$

and by (14) that

$$\text{Rank}(Q) = \text{Rank}(P) + 1 \ ,$$

and by (15) that

$$\text{Rank}(B) = \text{Rank}(A) - 1 \ .$$

$$BG = GB = (GB)^T ,$$

and (5) holds. Now

$$\begin{aligned}
BGB &= (AA^+ - A^+a\beta^{-1}a^TA^+)(A + ab^T + ba^T) \\
&= AA^+A + AA^+ab^T + AA^+ba^T - A^+a\beta^{-1}a^TA^+A \\
&\qquad - A^+a\beta^{-1}a^TA^+ab^T - A^+a\beta^{-1}a^TA^+ba^T \\
&= A + ab^T + ba^T - A^+a\beta^{-1}a^T + A^+a\beta^{-1}a^T ,
\end{aligned}$$

by (2), (37), (38) and (44),

$$= A + ab^T + ba^T \equiv B ,$$

and (2) is satisfied. And

$$\begin{aligned}
GBG &= (A^+A - A^+a\beta^{-1}a^TA^+)\Big(A^+ - A^+A^+a\beta^{-1}a^TA^+ \\
&\qquad - A^+a\beta^{-1}a^TA^+A^+ + A^+a\beta^{-2}(a^TA^+A^+A^+a)a^TA^+ \\
&\qquad - A^+\big(a(a^TA^+A^+b)\beta^{-1} - b\big)\gamma^{-1}\big(a(a^TA^+A^+b)\beta^{-1} - b\big)^TA^+\Big) \\
&= A^+AA^+ - A^+AA^+A^+a\beta^{-1}a^TA^+ - A^+AA^+a\beta^{-1}a^TA^+A^+ \\
&\qquad + A^+AA^+a\beta^{-2}(a^TA^+A^+A^+a)a^TA^+ \\
&\qquad - A^+AA^+\big(a(a^TA^+A^+b)\beta^{-1} - b\big)\gamma^{-1}\big(a(a^TA^+A^+b)\beta^{-1} - b\big)^TA^+ \\
&\qquad - A^+a\beta^{-1}a^TA^+A^+ + A^+a\beta^{-2}(a^TA^+A^+A^+a)a^TA^+ \\
&\qquad + A^+a\beta^{-2}a^TA^+A^+aa^TA^+A^+ - A^+a\beta^{-3}a^TA^+A^+a(a^TA^+A^+A^+a)a^TA^+ \\
&\qquad + A^+a\beta^{-1}a^TA^+A^+\big(a(a^TA^+A^+b)\beta^{-1} - b\big)\gamma^{-1}\big(a(a^TA^+A^+b)\beta^{-1} - b\big)^TA^+ \\
&= A^+ - A^+A^+a\beta^{-1}a^TA^+ - A^+a\beta^{-1}a^TA^+A^+ \\
&\qquad + A^+a\beta^{-2}(a^TA^+A^+A^+a)a^TA^+
\end{aligned}$$

$$
\begin{aligned}
BG = {} & (A + ab^T + ba^T)\big(A^+ - A^+A^{+T}a\beta^{-1}a^TA^+ - A^+a\beta^{-1}a^TA^+A^{+T} \\
& + A^+a\beta^{-2}(a^TA^+A^{+T}A^+a)a^TA^+ \\
& - A^+\big(a(a^TA^+A^{+T}b)\beta^{-1} - b\big)\gamma^{-1}\big(a(a^TA^+A^{+T}b)\beta^{-1} - b\big)^TA^+\big) \\
= {} & AA^+ - AA^+A^{+T}a\beta^{-1}a^TA^+ - AA^+a\beta^{-1}a^TA^+A^{+T} \\
& + AA^+a\beta^{-2}(a^TA^+A^{+T}A^+a)a^TA^+ \\
& - AA^+\big(a(a^TA^+A^{+T}b)\beta^{-1} - b\big)\gamma^{-1}\big(a(a^TA^+A^{+T}b)\beta^{-1} - b\big)^TA^+ \\
& + ab^TA^+ - ab^TA^+A^{+T}a\beta^{-1}a^TA^+ - ab^TA^+a\beta^{-1}a^TA^+A^{+T} \\
& + ab^TA^+a\beta^{-2}(a^TA^+A^{+T}A^+a)a^TA^+ \\
& - ab^TA^+\big(a(a^TA^+A^{+T}b)\beta^{-1} - b\big)\gamma^{-1}\big(a(a^TA^+A^{+T}b)\beta^{-1} - b\big)^TA^+ \\
& + ba^TA^+ - ba^TA^+A^{+T}a\beta^{-1}a^TA^+ - ba^TA^+a\beta^{-1}a^TA^+A^{+T} \\
& + ba^TA^+a\beta^{-2}(a^TA^+A^{+T}A^+a)a^TA^+ \\
& - ba^TA^+\big(a(a^TA^+A^{+T}b)\beta^{-1} - b\big)\gamma^{-1}\big(a(a^TA^+A^{+T}b)\beta^{-1} - b\big)^TA^+ \\
= {} & AA^+ - A^+a\beta^{-1}a^TA^+ + a(-\beta^{-1} + \beta^{-1})a^TA^+A^{+T} \\
& + a\big((a^TA^+A^{+T}b)^2\beta^{-2}\gamma^{-1}(1-1) + (a^TA^+A^{+T}b)\beta^{-1}(1-1)\big)a^TA^+ \\
& + a\big((a^TA^+A^{+T}b)\beta^{-1}\gamma^{-1}(1-1) + 1 - 1\big)b^TA^+ \\
& + b\big((a^TA^+A^{+T}b)\beta^{-1}\gamma^{-1}(1-1) + 1 - 1\big)a^TA^+ \\
& + b(\gamma^{-1} - \gamma^{-1})b^TA^+ \\
= {} & AA^+ - A^+a\beta^{-1}a^TA^+ \, , \qquad (45)
\end{aligned}
$$

which by (4) is symmetric. Note that $G$ is symmetric. Then by (13) and (45) it follows that

Theorem 3:

If

$$Pa = 0 , \tag{37}$$

$$Pb = 0 , \tag{38}$$

$$a^TA^+ab^TA^+b - (a^TA^+b+1)^2 = 0 , \tag{39}$$

$$a^TA^+a = 0 , \quad (b^TA^+b = 0) \tag{40}$$

and

$$b^TA^+b \neq 0 , \quad (a^TA^+a \neq 0) \tag{41}$$

then

$$\text{Rank}(B) = \text{Rank}(A) - 1 , \tag{42}$$

and

$$\begin{aligned} B^+ = A^+ &- A^+A^+a\beta^{-1}a^TA^+ - A^+a\beta^{-1}a^TA^+A^+ \\ &+ A^+a\beta^{-2}(a^TA^+A^+A^+a)a^TA^+ \\ &- A^+\left(a(a^TA^+A^+b)\beta^{-1} - b\right)\gamma^{-1}\left(a(a^TA^+A^+b)\beta^{-1} - b\right)^TA^+ \\ = G , \end{aligned} \tag{43}$$

where

$$\beta = a^TA^+A^+a \text{ , and}$$

and

$$\gamma = b^TA^+b .$$

Proof: Now $a^TA^+A^+a \neq 0$ for $a^TA^+A^+a = 0 \Rightarrow A^+a = 0 \Rightarrow Pa = a \neq 0$, a contradiction. Thus $a^TA^+A^+a$ has an inverse. Note also that $b^TA^+b \neq 0$ by assumption, and hence has an inverse and that by (39) and (40),

$$a^TA^+b = -1 . \tag{44}$$

Now

$$+ A^{+}(a,b)C^{-1}DC^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^{+} , \qquad \text{by (33)}$$

$$\equiv G .$$

Let $A = I - BG$ , which by (36)

$$= P + A^{+}(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^{+} .$$

Now by (34) it follows that

$$\text{Rank}\left(A^{+}(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^{+}\right) = 2 .$$

Hence by (14) we have

$$\text{Rank}(Q) = \text{Nullity}(B) \leqslant \text{Rank}(P) + 2 .$$

Now (24) and (25) imply that $Bx = 0$ for all $x$ satisfying $Ax = 0$ , which implies $\text{Rank}(Q) \geqslant \text{Rank}(P)$ . Now $B(A^{+}a) = 0$ .

<u>Proof:</u>

$$(A + AA^{+}ab^{T}A^{+}A + AA^{+}ba^{T}A^{+}A)A^{+}a = AA^{+}a + AA^{+}ab^{T}A^{+}a$$
$$AA^{+}ba^{T}A^{+}a ,$$

which by (27) and (33)

$$= AA^{+}a - AA^{+}a = 0 .$$

Similarly, $B(A^{+}b) = 0$ . So by (24), (25) and (34) it follows that

$$\text{Rank}(Q) = \text{Rank}(P) + 2$$

and then by (15) that

$$\text{Rank}(B) = \text{Rank}(A) - 2 .$$

$$
\begin{aligned}
GBG = \Bigg[ A^+ \quad & A^+A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+ - A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+A^+ \\
& + A^+(a,b)C^{-1}DC^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+ \Bigg] \cdot \Bigg[ AA^+ - A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+ \Bigg] \\
= A^+AA^+ & - A^+A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+ - A^+A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+AA^+ \\
& + A^+A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+ \\
& - A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+A^+AA^+ \\
& + A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+A^+A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+ \\
& + A^+(a,b)C^{-1}DC^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+AA^+ \\
& - A^+(a,b)C^{-1}DC^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+ \\
= A^+ & - A^+A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+ - A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+A^+ \\
& + A^+(a,b)C^{-1}DC^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+ \\
& + A^+(a,b)C^{-1}DC^{-1}\left[ I - \begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+(a,b) \right]\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+ ,
\end{aligned}
$$

by (3), (31) and (32),

$$
= A^+ - A^+A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+ - A^+(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix}A^+A^+
$$

So

$$BG = AA^{+} - A^{+}(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^{+} , \tag{36}$$

which is symmetric by (4).

Now

$$G^T = G \qquad \text{so by (13)}$$

$$BG = GB \qquad \text{and by (36)}$$

$$= (GB)^T = GB \qquad \text{and (5) holds.}$$

So

$$BGB = \left[ AA^{+} - A^{+}(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^{+} \right] \left[ A + (a,b)F\begin{pmatrix} a^T \\ b^T \end{pmatrix} \right]$$

$$= AA^{+}A + AA^{+}(a,b)F\begin{pmatrix} a^T \\ b^T \end{pmatrix} - A^{+}(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^{+}A$$

$$- A^{+}(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^{+}(a,b)F\begin{pmatrix} a^T \\ b^T \end{pmatrix}$$

$$= A + (a,b)F\begin{pmatrix} a^T \\ b^T \end{pmatrix} - A^{+}(a,b)C^{-1}\left[ I + \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^{+}(a,b)F \right] \begin{pmatrix} a^T \\ b^T \end{pmatrix} ,$$

by (2), (31) and (32),

$$= A + (a,b)F\begin{pmatrix} a^T \\ b^T \end{pmatrix} \qquad \text{by (35)}$$

$$\equiv B .$$

And

$$- a(\ +1)\alpha^{-1}\zeta^{-1}(\alpha + a^TA^+b + 1)b^TP$$

$$+ ba^TA^+ - b(a^TA^+a + a^TA^+b)\zeta^{-1}b^TP$$

$$- ba^TA^+ + b\zeta^{-1}(\alpha + a^TA^+b + 1)b^TP$$

$$= AA^+ + (I - AA^+)b\zeta^{-1}b^TP$$

$$+ a\Big(\zeta^{-1}\big(-1 + \alpha^{-1}(\alpha + a^TA^+b + 1)(1-1) - a^TA^+b - b^TA^+b$$

$$+ \alpha + b^TA^+b + 2\ a^TA^+b + 2 - \alpha - a^TA^+b - 1\big)\Big)b^TP$$

$$= AA^+ + Pb\zeta^{-1}b^TP\ , \tag{67}$$

which by (4) is symmetric, and which with (13) and the symmetry of $G$ and $B$ implies that $GB = (GB)^T$, and (5) holds. Now

$$BGB = (AA^+ + Pb\zeta^{-1}b^TP)(A + ab^T + ba^T)$$

$$= AA^+A + AA^+ab^T + AA^+ba^T + Pb\zeta b^TPba^T$$

$$= A + ab^T + (I - AA^+ + AA^+)ba^T\ ,$$

by (2), and (62),

$$= A + ab^T + ba^T$$

$$\equiv B\ .$$

And

$$GBG = (A^+A + Pb\zeta^{-1}b^TP)\Big(A^+ - A^+(a+b)\zeta^{-1}b^TP$$

$$- Pb\zeta^{-1}(a+b)^TA^+ + Pb\big((a+b)^TA^+(a+b) + 2\big)\zeta^{-2}b^TP$$

$$- \big(A^+a - Pb(\alpha + a^TA^+b + 1)\zeta^{-1}\big)\alpha^{-1}\ \big(a^TA^+ - \zeta^{-1}(\alpha + a^TA^+b + 1)b^TP\big)\Big)$$

$$= A^+AA^+ - A^+AA^+(a+b)\zeta^{-1}b^TP - A^+AA^+a\alpha^{-1}\big(a^TA^+ - \zeta^{-1}(\alpha + a^TA^+b + 1)b^TP\big)$$

$$- Pb\zeta^{-2}b^TPb(a+b)^TA^+ + Pb\zeta^{-3}b^TPb\big((a+b)^TA^+(a+b) + 2\big)b^TP$$

$$+ Pb\zeta^{-2}b^TPb(\alpha + a^TA^+b + 1)\alpha^{-1}\left(a^TA^+ - \zeta^{-1}(\alpha + a^TA^+b + 1)b^TP\right),$$

which by (2),

$$= A^+ - A^+(a+b)\zeta^{-1}b^TP - Pb\zeta^{-1}(a+b)^TA^+$$

$$+ Pb\zeta^{-2}\left((a+b)^TA^+(a+b) + 2\right)b^TP$$

$$- \left(A^+a - Pb(\alpha + a^TA^+b + 1)\zeta^{-1}\right)\alpha^{-1}\left(a^TA^+ - \zeta^{-1}(\alpha + a^TA^+b + 1)b^TP\right)$$

$$\equiv G .$$

Let $Q = I - BG$ . Then by (67),

$$Q = P - Pb\zeta^{-1}b^TP$$

and by (14),

$$\mathrm{Rank}(Q) \geq \mathrm{Rank}(P) - 1 . \tag{68}$$

We also have by (63) that

$$b \in \mathrm{Range}(B) \quad \text{but} \quad b \notin \mathrm{Range}(A) .$$

Note also that $x \in \mathrm{Range}(B)$ for all $x \in \mathrm{Range}(A)$. Hence

$$\mathrm{Rank}(B) \geq \mathrm{Rank}(A) + 1 .$$

But (15) and (68) give

$$\mathrm{Rank}(B) \leq \mathrm{Rank}(A) + 1 ,$$

so it follows that

$$\mathrm{Rank}(B) = \mathrm{Rank}(A) + 1 .$$

<u>Theorem 6</u>:

If

$$Pa = 0 , \tag{69}$$

$$Pb \neq 0 , \tag{70}$$

and

$$a^TA^+a = 0 \ , \tag{71}$$

then

$$\text{Rank}(B) = \text{Rank}(A) \ , \tag{72}$$

and

$$\begin{aligned} B^+ = D^+ &- D^+D^+v(v^TD^+D^+v)^{-1}v^TD^+ \\ &- D^+v(v^TD^+D^+v)^{-1}v^TD^+ \\ &+ D^+v(v^TD^+D^+v)^{-2}(v^TD^+D^+D^+v)v^TD^+ \\ &= G \ , \end{aligned} \tag{73}$$

where

$$u = a + b \ ,$$

$$v = a - b \ ,$$

$$D = A + u\,\frac{1}{2}\,u^T \ ,$$

and

$$\begin{aligned} D^+ = A^+ &- Pb(b^TPb)^{-1}u^TA^+ - A^+u(b^TPb)^{-1}b^TA^+ \\ &+ Pb(b^TPb)^{-2}(u^TA^+u + 2)b^TP = C \ . \end{aligned} \tag{74}$$

<u>Proof</u>: Note first the following:

(i) From (70) it follows that $b^TPb = b^TPPb > 0$ and thus has an inverse.

(ii) $DC = \left[A + u\,\frac{1}{2}\,w^T\right]\left(A^+ - Pb(b^TPb)^{-1}u^TA^+ - A^+u(b^TPb)^{-1}b^TP\right.$

$\left. + Pb(b^TPb)^{-2}(u^TA^+u + 2)b^TP\right)$

$= AA^+ - AA^+u(b^TPb)^{-1}b^TA^+ + u\,\frac{1}{2}\,u^TA^+$

$- u\,\frac{1}{2}\,u^TPb(b^TPb)^{-1}u^TA^+ - u\,\frac{1}{2}\,u^TA^+u(b^TPb)b^TP$

$$+ u \frac{1}{2} u^T Pb(b^T Pb)^{-2}(u^T A^+ u + 2) b^T P$$

$$= AA^+ - AA^+ u(b^T Pb)^{-1} b^T P$$

$$- u \frac{1}{2} u^T A^+ u(b^T Pb)^{-1} b^T P + u \frac{1}{2} (u^T A^+ u + 2)(b^T Pb)^{-1} b^T P$$

$$= AA^+ + (I - AA^+) u(b^T Pb)^{-1} b^T P$$

$$= AA^+ + Pb(b^T Pb)^{-1} b^T P \ , \qquad (75)$$

which by (5) is symmetric. Now from $C = C^T$ and $D = D^T$, (13) and (75), it follows that $DC = CD = (CD)^T$ and (4) holds. Then

$$DCD = \left(AA^+ - Pb(b^T Pb)^{-1} b^T P\right)(A + u \frac{1}{2} u^T)$$

$$= AA^+ A + AA^+ u \frac{1}{2} u^T - Pb(b^T Pb)^{-1} b^T Pu \frac{1}{2} u^T \ ,$$

which by (2) and (69)

$$= A + AA^+ u \frac{1}{2} u^T - Pb \frac{1}{2} u^T$$

$$= A + (I - AA^+ + AA^+) u \frac{1}{2} u^T \ ,$$

by (69)

$$= A + u \frac{1}{2} u^T \equiv D \ .$$

And

$$CDC = [A^+ - Pb(b^T Pb)^{-1} u^T A^+ - A^+ u(b^T Pb)^{-1} b^T A^+$$

$$+ Pb(b^T Pb)^{-2}(u^T A^+ u + 2) b^T P][AA^+ - Pb(b^T Pb)^{-1} b^T P]$$

$$= A^+ AA^+ - Pb(b^T Pb)^{-1} u^T A^+ AA^+ - A^+ u(b^T Pb)^{-1} b^T A^+ AA^+$$

$$- Pb(b^T Pb)^{-2}(u^T A^+ u + 2) b^T Pb(b^T Pb)^{-1} b^T P \ ,$$

which by (3)

$$= A^+ - Pb(b^TPb)^{-1}u^TA^+ - A^+u(b^TPb)^{-1}b^TA^+$$

$$- Pb(b^TPb)^{-2}(u^TA^+u + 2)b^TP \equiv C .$$

Thus $C = D^+$ . Let $R = I - DD^+$ . Then by (75)

$$R = P - Pb(b^TPb)^{-1}b^TP ,$$

and by the same arguments as in Theorem 5,

$$\text{Rank}(D) = \text{Rank}(A) + 1 .$$

(iii) $(I - DD^+)v = v - AA^+v + Pb(b^TPb)^{-1}b^TPv$

$$= (I - AA^+ - I + AA^+)v = 0 , \qquad (76)$$

so $v^TD^+D^+v \neq 0$ , since

$$D^+v = 0 \Rightarrow (I - DD^+)v = v \neq 0 ,$$

a contradiction. Thus $v^TD^+D^+v$ has an inverse.

(iv) $v^TD^+v = v^TA^+v + 2\,u^TA^+v + u^TA^+u = 4a^TA^+a + 2 = 2 \qquad (77)$

Now

$$BG = \left(A + u\,\tfrac{1}{2}\,u^T - v\,\tfrac{1}{2}\,v^T\right)\Big(D^+ - D^+D^+v(vD^+D^+v)^{-1}v^TD^+$$

$$- D^+v(v^TD^+D^+v)^{-1}v^TD^+D^+ + D^+v(v^TD^+D^+v)^{-2}(v^TD^+D^+D^+v)v^TD^+\Big)$$

$$= D - v\,\tfrac{1}{2}\,v^T\Big(D^+ - D^+D^+v(vD^+D^+v)^{-1}v^TD^+$$

$$- D^+v(v^TD^+D^+v)^{-1}v^TD^+D^+ + D^+v(v^TD^+D^+v)^{-2}(v^TD^+D^+D^+v)v^TD^+\Big)$$

$$= DD^+ - DD^+D^+v(v^TD^+D^+v)^{-1}v^TD^+$$

$$- DD^+v(v^TD^+D^+v)^{-1}v^TD^+D^+ + DD^+v(v^TD^+D^+v)^{-2}(v^TD^+D^+D^+v)v^TD^+$$

$$- v\,\tfrac{1}{2}\,v^TD^+ + v^T\,\tfrac{1}{2}\,(v^TD^+D^+v)(v^TD^+D^+v)^{-1}v^TD^+$$

$$+ v \frac{1}{2} (v^T D^+ v)(v^T D^+ D^+ v)^{-1} v^T D^+ D^+$$

$$- v \frac{1}{2} (v^T D^+ v)(v^T D^+ D^+ v)^{-2} (v^T D^+ D^+ D^+ v) v^T D^+$$

$$= DD^+ - D^+ v (v^T D^+ D^+ v)^{-1} v^T D^+$$

$$- v (v^T D^+ D^+ v)^{-1} v^T D^+ D^+ + v (v^T D^+ D^+ v)^{-2} (v^T D^+ D^+ D^+ v) v^T D^+$$

$$- v \frac{1}{2} v^T D^+ + v^T \frac{1}{2} v^T D^+ + v (v^T D^+ D^+ v)^{-1} v^T D^+ D^+$$

$$- v (v^T D^+ D^+ v)^{-2} (v^T D^+ D^+ D^+ v) v^T D^+$$

$$= DD^+ - D^+ v (v^T D^+ D^+ v)^{-1} v^T D^+ , \tag{78}$$

which by (4) is symmetric. Since $B = B^T$ , $G = G^T$ and $BG = (BG)^T$ , then by (13) it follows that $GB = BG = (GB)^T$ , and (5) holds. Then

$$BGB = \left(DD^+ - D^+ v (v^T D^+ D^+ v)^{-1} v^T D^+\right)(D - v \frac{1}{2} v^T)$$

$$= DD^+ D - DD^+ v \frac{1}{2} v^T - D^+ v (v^T D^+ D^+ v)^{-1} v^T D^+ D$$

$$+ D^+ v (v^T D^+ D^+ v)^{-1} (v^T D^+ v) \frac{1}{2} v^T$$

$$= D - v \frac{1}{2} v^T$$

$$\equiv B .$$

And

$$GBG = [D^+ D - D^+ v (v^T D^+ D^+ v)^{-1} v^T D^+][D^+ - D^+ D^+ v (v^T D^+ D^+ v)^{-1} v^T D^+$$

$$- D^+ v (v^T D^+ D^+ v)^{-1} v^T D^+ D^+$$

$$+ D^+ v (v^T D^+ D^+ v)^{-2} (v^T D^+ D^+ D^+ v) v^T D^+]$$

$$= D^+ DD^+ - D^+ DD^+ D^+ v (v^T D^+ D^+ v)^{-1} v^T D^+$$

$$- D^+DD^+v(v^TD^+D^+v)^{-1}v^TD^+D^+$$

$$+ D^+DD^+v(v^TD^+D^+v)^{-2}(v^TD^+D^+D^+v)v^TD^+$$

$$- D^+v(v^TD^+D^+v)^{-1}v^TD^+D^+$$

$$+ D^+v(v^TD^+D^+v)^{-2}(v^TD^+D^+D^+v)v^TD^+$$

$$+ D^+v(v^TD^+D^+v)^{-2}(v^TD^+D^+v)v^TD^+D^+$$

$$- D^+v(v^TD^+D^+v)^{-3}(v^TD^+D^+v)(v^TD^+D^+D^+v)v^TD^+$$

$$= D^+ - D^+D^+v(v^TD^+D^+v)^{-1}v^TD^+$$

$$- D^+v(v^TD^+D^+v)^{-1}v^TD^+D^+$$

$$+ D^+v(v^TD^+D^+v)^{-2}(v^TD^+D^+D^+v)v^TD^+$$

$$\equiv G .$$

Let $Q = I - BG$ . Then by (78) $Q = R + D^+v(v^TD^+D^+v)^{-1}v^TD^+$ , which with (15) implies that

$$\text{Rank}(Q) \leqslant \text{Rank}(R) + 1 .$$

Now $B(D^+v) = (D - v\,\frac{1}{2}\,v^T)D^+v$ , which by (76) and (77)

$$= v - v = 0 .$$

It also follows from (76) that $Bx = 0$ for all $x$ satisfying $Dx = 0$ and

$$D(D^+v) = v \neq 0 .$$

Hence

$$\text{Rank}(Q) = \text{Rank}(R) + 1$$

which with (15) implies

$$\text{Rank}(B) = \text{Rank}(D) - 1 = \text{Rank}(A) .$$

Theorem 7:

If

$$Pa \neq 0 \ , \tag{79}$$

$$Pb \neq 0 \ , \tag{80}$$

and

$$Pa \neq Pbk \quad \text{for any} \quad k \neq 0 \ , \tag{81}$$

then

$$\text{Rank}(B) = \text{Rank}(A) + 2 \tag{82}$$

and

$$B^+ = A^+ - P(a,b)\left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} P(a,)\right]^{-1} \begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+$$

$$- A^+(a,b) \left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} P(a,b)\right]^{-1} \begin{pmatrix} a^T \\ b^T \end{pmatrix} P$$

$$+ P(a,b) \left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} P(a,b)\right]^{-1} \left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] \left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} P(a,b)\right]^{-1} \begin{pmatrix} a^T \\ b^T \end{pmatrix} P$$

$$= G \ . \tag{83}$$

Proof: Let

$$C = \left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} P(a,b)\right] \ , \quad F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$D = \left[\begin{pmatrix} a^T \\ b^T \end{pmatrix} A^+(a,b) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] \ .$$

Note first that (79), (80) and (81) imply that $\det C = a^T Pa \; b^T Pb - (a^T Pb)^2 \neq 0$ and hence $C$ has an inverse. Now

$$BG = \left[A + (a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}\right]\left[A^+ - P(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+\right.$$

$$\left. - A^+(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P + P(a,b)C^{-1}DC^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P\right]$$

$$= AA^+ - AA^+(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P$$

$$+ (a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+ - (a,b)F\left[\begin{pmatrix}a^T\\b^T\end{pmatrix}P(a,b)\right]C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+$$

$$- (a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P + (a,b)F\left[\begin{pmatrix}a^T\\b^T\end{pmatrix}P(a,b)\right]C^{-1}DC^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P$$

$$= AA^+ - AA^+(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P + (a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+$$

$$- (a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+ - (a,b)F(D-F)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P$$

$$+ (a,b)F\ D\ C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P$$

$$= AA^+ - AA^+(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P + (a,b)F\ F\ C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P$$

$$= AA^+ + (I-AA^+)(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P\ ,$$

since $FF = I$

$$= AA^+ + P(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P\ , \qquad (84)$$

which by (4) and the symmetry of $C$ is symmetric. Now since $G$ is symmetric, then by (13) and (56)

$$BG = GB = (GB)^T \quad \text{so (5) holds.}$$

So

$$BGB = \left[AA^+ + P(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P\right]\left[A + (a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}\right]$$

$$= AA^+A + AA^+(a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix} + P(a,b)C^{-1}\left[\begin{pmatrix}a^T\\b^T\end{pmatrix}P(a,b)\right]F\begin{pmatrix}a^T\\b^T\end{pmatrix}$$

$$= A + AA^+(a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix} + P(a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}$$

$$= A + (AA^+ + I - AA^+)(a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}$$

$$= A + (a,b)F\begin{pmatrix}a^T\\b^T\end{pmatrix}$$

$$\equiv B\ .$$

And

$$GBG = \left[A^+A + P(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P\right]$$

$$\cdot\left[A^+ - P(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+ - A^+(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P + P(a,b)C^{-1}DC^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P\right]$$

$$= A^+AA^+ - A^+AA^+(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P - P(a,b)C^{-1}\left[\begin{pmatrix}a^T\\b^T\end{pmatrix}P(a,b)\right]C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+$$

$$+ P(a,b)C^{-1}\left[\begin{pmatrix}a^T\\b^T\end{pmatrix}P(a,b)\right]C^{-1}DC^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P$$

$$= A^+ - A^+(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P - P(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}A^+ + P(a,b)C^{-1}DC^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P$$

$$\equiv G\ .$$

Let $Q = I - BG$ . Then by (84)

$$Q = P - P(a,b)C^{-1}\begin{pmatrix}a^T\\b^T\end{pmatrix}P\ .$$

From (81) it follows that

$$\text{Rank}\left[P(a,b)C^{-1}\begin{pmatrix} a^T \\ b^T \end{pmatrix} P\right] = 2$$

and then by (14)

$$\text{Rank}(Q) \geqslant \text{Rank}(P) - 2 \ .$$

Now $Px = 0 \Rightarrow Qx = 0$ . Note also by (79) and (80)

$$P \cdot Pa = Pa \neq 0$$

$$P \cdot Pb = Pb \neq 0$$

but

$$QPa = Pa - P(a,b)C^{-1}\begin{pmatrix} a^T Pa \\ b^T Pb \end{pmatrix}$$

$$= Pa - (Pa,Pb)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \ ,$$

and similarly

$$QPb = 0 \ .$$

Then by (81) it follows that

$$\text{Nullity}(Q) \geqslant \text{Nullity}(P) + 2 \ ,$$

which with (14) implies

$$\text{Rank}(B) = \text{Rank}(B) + 2 \ .$$

<u>Theorem 8</u>:

If

$$Pa \neq 0 \ , \tag{85}$$

$$Pb \neq 0 \ , \tag{86}$$

$$Pa = Pbk \quad \text{for some} \quad k \neq 0 \ , \tag{87}$$

and

$$(a-bk)^T A^+ (a-bk) - 2k \neq 0 \tag{88}$$

then

$$\mathrm{Rank}(B) = \mathrm{Rank}(A) + 1 \tag{89}$$

and

$$\begin{aligned} B^+ = {} & A^+ - A^+ v \lambda^{-1} v^T A^+ - Pa\psi^{-1}\lambda^{-1}(a\tau + bk\omega)^T A^+ \\ & - A^+(a\tau + bk\omega)\psi^{-1}\lambda^{-1} a^T P \\ & + Pa\psi^{-2}\lambda^{-1}k^2\left(a^T A^+ a b^T A^+ b - (a^T A^+ b + 1)^2\right) a^T P \\ = {} & G , \end{aligned} \tag{90}$$

where

$$k = \frac{a^T Pa}{a^T Pb} = \frac{a^T Pb}{b^T Pb} ,$$

$$v = a - bk ,$$

$$\lambda = v^T A^+ v - 2k ,$$

$$\psi = a^T Pa ,$$

$$\tau = k^2 b^T A^+ b - k(a^T A^+ b + 1) ,$$

and

$$\omega = a^T A^+ a - k(a^T A^+ b + 1) .$$

<u>Proof</u>: Let $u = (a + bk)$. Note the following:

(i) From (85) it follows that $a^T Pa = a^T PPa > 0$ and from (88) it follows that $\lambda \neq 0$ and so each has an inverse.

(ii) $A^+(a\tau + bk\omega)2$

$$\begin{aligned} & = A^+\left(a\left(k^2 b^T A^+ b - k(a^T A^+ b + 1)\right) + bk\left(a^T A^+ a - k(a^T A^+ b + 1)\right)\right)2\lambda^{-1} \\ & = A^+\left(a\left(1 - \lambda^{-1}(a^T A^+ a - k^2 b^T A^+ b)\right) + bk\left(1 + \lambda^{-1}(a^T A^+ a - k^2 b^T A^+ b)\right)\right) \\ & = A^+ u - A^+ v\lambda^{-1} v^T A^+ u = (A^+ - A^+ v \lambda^{-1} v^T A^+) u . \end{aligned} \tag{91}$$

(iii) From (87) it follows that

$$v^TP = 0 \ ,$$

$$u^TP = 2a^TP \ ,$$

and

$$AA^+v = A^+Av = v \ . \tag{92}$$

(iv) $(A - v\,\frac{1}{2k}\,v^T)(A^+ - A^+v\lambda^{-1}v^TA^+)$

$= AA^+ - v\lambda^{-1}v^TA^+ - v\,\frac{1}{2k}\,v^TA^+ + v\,\frac{1}{2k}\,(v^TA^+v)\lambda^{-1}v^TA^+$

$= AA^+ - v(\lambda^{-1} + \frac{1}{2k} - \frac{1}{2k}\,\lambda^{-1}v^TA^+v)v^TA^+$

$= AA^+ + v\left(\lambda^{-1}\,\frac{1}{2k}\,(v^TA^+v - 2k - \lambda)\right)v^TA^+$

$$= AA^+ \ . \tag{93}$$

(v) $\lambda u^TA^+u - (u^TA^+v)^2 + 2k\lambda$

$= (a^TA^+a + k^2b^TA^+b)^2 - 4k^2(a^TA^+b)^2 - 2k(a^TA^+a + k^2b^TA^+b + 2ka^TA^+b)$

$\quad - (a^TA^+a)^2 - (k^2b^TA^+b)^2 + 2k^2a^TAab^TA^+b$

$\quad + 2k(a^TA^+a + k^2b^TA^+b - 2ka^TA^+b) - 4k^2$

$= 4k^2\left(a^TA^+ab^TA^+b - (a^TA^+b)^2 - 2a^TA^+b - 1\right)$

$$= 4k^2\left(a^TA^+ab^TA^+b - (a^TA^+b + 1)^2\right) \ . \tag{94}$$

Now

$BG = [A - v\,\frac{1}{2k}\,v^T + u\,\frac{1}{2k}\,u^T][A^+ - A^+v\lambda^{-1}v^TA^+$

$\quad - Pa\psi^{-1}\lambda^{-1}(a\tau + bk\omega)^TA^+ - A^+(a\tau + bk\omega)\psi^{-1}\lambda^{-1}a^TP$

$\quad + Pa\psi^{-2}\lambda^{-1}k^2\left(a^TA^+ab^TA^+b - (a^TA^+b + 1)^2\right)a^TP]$

$$= (A - v\,\frac{1}{2k}\,v^T)(A^+ - A^+v\lambda^{-1}v^TA^+)$$

$$- (A - v\,\frac{1}{2k}\,v^T)(A^+ - A^+v\lambda^{-1}v^TA^+)u\,\frac{1}{2}\,\psi^{-1}a^TP$$

$$+ u\,\frac{1}{2k}\,u^TA^+ - u\,\frac{1}{2k}\,u^TA^+v\lambda^{-1}v^TA^+$$

$$- u\,\frac{1}{2k}\,u^TPa\psi^{-1}\,\frac{1}{2}\,u^TA^+$$

$$+ u\,\frac{1}{2k}\,u^TPa\,\psi^{-1}\frac{1}{2}\,u^TA^+v\lambda^{-1}v^TA^+$$

$$- u\,\frac{1}{2k}\,(u^TA^+u)\,\frac{1}{2}\,\psi^{-1}a^TP$$

$$+ u\,\frac{1}{2k}\,(u^TA^+v)^2\lambda^{-1}\,\frac{1}{2}\,\psi^{-1}a^TP$$

$$+ u\,\frac{1}{2k}\,u^TPa\psi^{-2}\lambda^{-1}k^2\left(a^TA^+ab^TA^+b - (a^TA^+b + 1)^2\right)a^TP\ ,$$

which by (93) and (94)

$$= AA^+ - AA^+u\,\frac{1}{2}\,\psi^{-1}a^TP$$

$$+ u\left(\frac{1}{2}\,\psi^{-1}\,\frac{1}{2k}\,\lambda^{-1}\left(-\lambda u^TA^+u + (u^TA^+v)^2\right.\right.$$

$$\left.\left. + 4k^2\,a^TA^+ab^TA^+b - (a^TA^+b + 1)^2\right)\right)a^TP$$

$$= AA^+ - AA^+u\,\frac{1}{2}\,\psi^{-1}a^TP$$

$$+ u\left(\frac{1}{2}\,\psi^{-1}\,\frac{1}{2k}\,\lambda^{-1}\left(-\lambda u^TA^+u + (u^TA^+v)^2 + \lambda u^TA^+u\right.\right.$$

$$\left.\left. - (u^TA^+v)^2 + 2k\lambda\right)\right)a^TP$$

$$= AA^+ - AA^+u\,\frac{1}{4}\,\psi^{-1}u^TP + u\,\frac{1}{4}\,\psi^{-1}u^TP$$

$$= AA^+ + Pu(u^TPu)^{-1}u^TP\ , \tag{95}$$

which by (4) is symmetric.

$$D = A + u \frac{1}{2k} u^T ,$$

$$\text{Rank}(D) = \text{Rank}(A) ,$$

and $D^+$ is given by Formula (106). Now by (19), $\text{Rank}(B) = \text{Rank}(A)$, which will occur if and only if

$$(I - DD^+)v = 0 \quad \text{and}$$

$$v^T D^+ v \neq -2k .$$

By the interlocking eigenvalue property, $B$ will remain positive semidefinite if and only if

$$v^T D^+ v < 2k \Leftrightarrow 4k^2[a^T A^+ a b^T A^+ b - (a^T A^+ b + 1)^2] < 0$$

$$\Leftrightarrow a^T A^+ a b^T A^+ b - (a^T A^+ b + 1)^2 < 0 .$$

Corollary 2:

If

$$Pa = 0$$

and

$$Pb \neq 0 ,$$

then $B$ will be indefinite.

Proof: By Lemma 1, $a^T A^+ a > 0$ so the conditions of Theorem 5 hold. Define

$$u = a + kb$$

$$v = a - kb$$

for some $k > 0$. Then $u \frac{1}{2k} u^T$ is a positive dyad, $Pu \neq 0$, $D = A + u \frac{1}{2k} u^T$ will be positive semidefinite, and $\text{Rank}(D) = \text{Rank}(A) + 1$, which will occur if and only if

$u \frac{1}{2k} u^T$ is a positive dyad and $D = A + u \frac{1}{2k} u^T$ will be positive semidefinite. Note also that $1 + \frac{1}{2k} u^T A^+ u > 0$, which with Theorem 10 implies that $\mathrm{Rank}(D) = \mathrm{Rank}(A)$ and $D^+$ is given by Formula (106). By (51), $\mathrm{Rank}(B) = \mathrm{Rank}(A) - 1$, which will occur if and only if

$$(I - D^+D)v = 0 \quad \text{and}$$

$$v^T D^+ v = -2k \ .$$

Now

$$
\begin{aligned}
v^T D^+ v \;&=\; v^T A^+ v - (v^T A^+ u)^2 (2k + w^T A^+ u)^{-1} = 2k \\
&\Longleftrightarrow (v^T A^+ v - 2k)(2k + u^T A^+ u) - (v^T A^+ u)^2 = 0 \\
&\Longleftrightarrow 4k^2(-2a^T A^+ b - 1) + (a^T A^+ a + k^2 b^T A^+ b)^2 \\
&\qquad - (2ka^T A^+ b)^2 - (a^T A^+ a - k^2 b^T A^+ b)^2 = 0 \\
&\Longleftrightarrow 4k^2[a^T A^+ a b^T A^+ b - (a^T A^+ b + 1)^2] = 0 \\
&\Longleftrightarrow a^T A^+ a b^T A^+ b - (a^T A^+ b + 1)^2 = 0 \ .
\end{aligned}
$$

If $k < 0$, the roles of $u \frac{1}{2k} u^T$ and $v - \frac{1}{2k} v^T$ as positive and negative dyads are reversed, and the same result obtains.

(ii) Suppose $a^T A^+ a b^T A^+ b - (a^T A^+ b + 1)^2 < 0$. Then the conditions of Theorem 1 hold. Define

$$u = (a + bk) \quad \text{and}$$

$$v = (a - bk) \quad \text{for some} \quad k > 0 \ .$$

As in part (i), $u \frac{1}{2k} u^T$ is a positive dyad, $Pu = 0$, $1 + \frac{1}{2k} u^T A^+ u > 0$, and for

Now $Pa = 0$ , $a \neq 0 \Rightarrow a = E^*w$ , where $E^* = [e_1,\ldots,e_k]$ is the $n \times k$ matrix of eigenvectors of $A$ associated with nonzero eigenvalues, and $w$ is the $k \times 1$ vector whose ith element is $\lambda_i \cdot e_i^T a$ , $i = 1,\ldots,k$ . Then

$$\begin{aligned} a^T A^+ a &= w^T (E^*)^T E \lambda^+ E^T E^* w \\ &= w^T [I_k, 0] \lambda^+ [I_k, 0]^T w \\ &= w^T (\lambda^*)^+ w \ , \end{aligned}$$

where $\lambda^*$ is the positive definite $k \times k$ diagonal matrix of nonzero eigenvalues of $A$ , giving

$$a^T A^+ a > 0 \ .$$

Corollary 1:

If

$$Pa = 0 \ ,$$

and

$$Pb = 0$$

then a necessary and sufficient condition for $B$ to be positive semidefinite is that

$$a^T A^+ a b^T A^+ b - (a^T A^+ b + 1)^2 \leqslant 0 \ .$$

Proof: There are two cases.

(i) Suppose $a^T A^+ a b^T A^+ b - (a^T A^+ b + 1)^2 = 0$ . Then by Lemma 1, the conditions of Theorem 4 hold. That is, conditions of Theorems 2 and 3 cannot hold for $A$ positive semidefinite.

Define $u$ , $v$ , and $k$ as in Theorem 4. If $k > 0$ , then

will equal the rank of $A$. In either of the above cases, $B$ will remain positive semidefinite.

For values of $c$ less than the above, that is, if $c < -(a^T A^+ a)^{-1}$, the new eigenvalue will shift below zero, rendering $B$ indefinite.

Suppose, as in the process of building an estimate of the positive part of a matrix, it is desired that $B$ remain positive semidefinite. Then by the reasoning above, we see that in the case that $c < -(a^T A^+ a)^{-1}$, a fraction of the dyad, namely $ada^T$ for any $-(a^T A^+ a)^{-1} \leq d < 0$, could still be added to $A$, and $B$ would remain positive semidefinite.

It will next be useful to examine the effect on the definiteness of a matrix of a rank two perturbation of the form $(ab^T + ba^T)$. This effect can readily be demonstrated by regarding $(ab^T + ba^T)$ as the successive sum of two -- one positive and one negative -- symmetric rank one perturbations.

As before, let $A$ be an $n \times n$ symmetric positive semidefinite matrix and let $a$ and $b$ be $n \times 1$ vectors. Let $B = A + ab^T + ba^T$ and $P = I - A^+A$.

<u>Lemma 1</u>:

If $Pa = 0$, $a \neq 0$, then $a^T A^+ a > 0$.

<u>Proof</u>: $A$ positive semidefinite implies $A = E\lambda E^T$, where $E$ is the $n \times n$ matrix of eigenvectors of $A$, $E^T E = I$, and $\lambda$ is the $n \times n$ diagonal matrix of eigenvalues of $A$, where the $\lambda_i$'s are ordered so that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

(i) If $c$ is a positive scalar, then depending on whether $Pa = 0$ or not, the rank of $B = A + aca^T$ will remain the same or increase by one. In either case the eigenvalues of $A$ will shift in a positive direction and $B$ will remain positive semidefinite.

(ii) If $c$ is a negative scalar and if $Pa \neq 0$ (this can only happen if $A$ is not of full rank and therefore has at least one zero eigenvalue), then by the interlocking eigenvalue property one of the zero eigenvalues of $A$ will decrease and become negative, leaving $B$ indefinite.

(iii) If $c$ is a negative scalar and if $Pa = 0$ , then

$$B = A + AA^{+}aca^{T}$$

and all elements of the null space of $A$ are elements of the null space of $B$ . In particular, all eigenvectors associated with the zero eigenvalue for $A$ , if any, will also be so for $B$ . Now for $a \neq 0$ , there exists some smallest eigenvalue of $A$ of multiplicity $\eta$ whose associated eigenvectors are not all orthogonal to $a$ . Then by the interlocking eigenvalue property, this eigenvalue (or one of these if $\eta > 1$) will decrease to a value $\beta$ which solves $a^T E(\lambda - \beta I)^{+} E^T a = -\frac{1}{c}$ , where $A = E\lambda E^T$ is the eigenvalue-eigenvector decomposition of $A$ . In particular, if the scalar $c$ satisfies $c = -(a^T A^{+} a)^{-1}$ , then the new eigenvalue $\beta$ will be zero and the rank of $B$ will be one less than the rank of $A$ . For values of $c$ greater than the above, the new eigenvalue will be positive and the rank of $B$

(ii) If

$$Pa = 0$$

and

$$1 + ca^T A^+ a = 0 ,$$

then

$$\text{Rank}(B) = \text{Rank}(A) - 1$$

and

$$B^+ = A^+ - A^+ A^+ a\beta^{-1} a^T A^+ - A^+ a\beta^{-1} a^T A^+ A^+ + A^+ a\beta^{-2}(a^T A^+ A^+ A^+ a) a^T A^+ , \tag{107}$$

where $\beta = a^T A^+ A^+ a$ .

(iii) If

$$Pa \neq 0 ,$$

then

$$\text{Rank}(B) = \text{Rank}(A) + 1$$

and

$$B^+ = A^+ - Pa\gamma^{-1} a^T A^+ - A^+ a\gamma^{-1} a^T P + Pa\gamma^{-2}(c^{-1} + a^T A^+ a) a^T P , \tag{108}$$

where $\gamma = a^T Pa$ .

Following the above results and the interlocking eigenvalue theorem [see McCormick (1983)] we can make the following observations:

Suppose $A$ is an $n \times n$ symmetric, positive semidefinite matrix which is perturbed by a dyad of the form $aca^T$ where $a$ and $c$ are defined as in Theorem 10. Let $P = I - AA^+$ .

## 3. Computing an Estimate of the Positive Part of a Symmetric Matrix and Directions of Nonpositive and Negative Curvature

The results of the previous section can be directly applied in determining the definiteness of a matrix $A$ which is given by (1), and in turn computing an estimate of the positive part of $A$ and its associated directions of nonpositive and negative curvature.

It will first be useful to state the following results concerning symmetric rank one perturbations due to Meyer (1973) and McCormick (1976). Statements and notation are as in McCormick (1976).

Theorem 10:

Let $A$ be an $n \times n$ symmetric matrix, $a$ an $n \times 1$ vector, and $c$ a nonzero scalar. Define

$$P = I - AA^+ \text{ and}$$

$$B = A + aca^T .$$

(i) If

$$Pa = 0$$

and

$$1 + ca^TA^+a \neq 0 ,$$

then

$$\text{Rank}(B) = \text{Rank}(A)$$

and

$$B^+ = A^+ - A^+a\left(c^{-1} + a^TA^+a\right)^{-1}a^TA^+ . \tag{106}$$

Let $Q = I - BG$ . Then by (105)

$$Q = R + D^+ v(v^T D^+ D^+ v)^{-1} v^T D^+$$

and

$$\mathrm{Rank}(Q) \leqslant \mathrm{Rank}(R) + 1$$

by (14). It follows from (104) that $Bx = 0$ for all $x$ satisfying $Dx = 0$ . Hence

$$\mathrm{Rank}(Q) \geqslant \mathrm{Rank}(R) \ .$$

Now $B(D^+ v) = (D - v \frac{1}{2k} v^T) D^+ v$

$= DD^+ v - v \frac{1}{2k} v^T D^+ v$ , which by (99) and (102)

$= v - v = 0$ .

But $DD^+ v = v \neq 0$ by (102), so we have

$$\mathrm{Rank}(Q) = \mathrm{Rank}(R) + 1$$

which with (15) gives

$$\mathrm{Rank}(B) = \mathrm{Rank}(D) - 1 = \mathrm{Rank}(A) \ .$$

$$\text{Nullity}(R) \geqslant \text{Nullity}\ (P) + 1$$

and

$$\text{Rank}(R) = \text{Rank}\ P - 1$$

or

$$\text{Rank}(D) = \text{Rank}(A) + 1\ .$$

(iv) $$(I - DD^+)v = v - AA^+v + Pa(a^TPa)^{-1}a^TPv = v - v = 0\ , \tag{104}$$

by (102). So $v^TD^+D^+v \neq 0$ since $D^+V = 0 \Rightarrow (I - DD^+)v = v \neq 0$, a contradiction. Hence $v^TD^+D^+v$ has an inverse. Now

$$\begin{aligned}
BG &= [A + u^T \frac{1}{2k} u - v \frac{1}{2k} v^T] \cdot G \\
&= [D - v \frac{1}{2k} v^T][D^+ - D^+D^+v(v^TD^+D^+v)^{-1}v^TD^+ \\
&\quad - D^+v(v^TD^+D^+v)^{-1}v^TD^+D^+ \\
&\quad + D^+v(v^TD^+D^+v)^{-2}(v^TD^+D^+D^+v)v^TD^+] \\
&= DD^+ - D^+v(v^TD^+D^+v)^{-1}v^TD^+\ ,
\end{aligned} \tag{105}$$

by the same process as in Theorem 6, and

$$BG = (BG)^T$$

by (4). So we have by $B = B^T$, $G = G^T$, $BG = (BG)^T$ and (13) that $GB = BF = (GB)^T$. And by the same arguments as in Theorem 6, we have $BGB = D - v \frac{1}{2k} v^T \equiv B$, and

$$\begin{aligned}
GBG &= D^+ - D^+D^+v(v^TD^+D^+v)^{-1}v^TD^+ \\
&\quad - D^+v(v^TD^+D^+v)^{-1}v^TD^+D^+ \\
&\quad + D^+v(v^TD^+D^+v)^{-2}(v^TD^+D^+D^+v)v^TD^+ \\
&\equiv G\ .
\end{aligned}$$

$$= A + u \frac{1}{2k} u^T (AA^+ + I - AA^+) ,$$

by (2), (9), and (98),

$$= A + u \frac{1}{2k} u^T \equiv D ,$$

and

$$CDC = \left(AA^+ + Pa(a^TPa)^{-1}a^TP\right)\Big[A^+ - Pa(2a^TPa)^{-1}u^TA^+$$

$$- A^+u(2a^TPa)^{-1}a^TP$$

$$+ Pa\left(k(a^TA^+b + 1)\right)(a^TPa)^{-2}a^TP\Big]$$

$$= AA^+A^+ - AA^+A^+u(2a^TPa)^{-1}a^TP$$

$$- Pa(a^TPa)^{-1}a^TPa(2a^TPa)^{-1}u^TA^+$$

$$+ Pa(a^TPa)^{-1}a^TPa\left(k(a^TA^+b + 1)\right)(a^TPa)^{-2}a^TP ,$$

which by (3), and (9)

$$= A^+ - A^+u(2a^TPa)^{-1}a^TP - Pa(2a^TPa)^{-1}u^TA^+$$

$$+ Pa\left(k(a^TA^+b + 1)\right)(a^TPa)^{-2}a^TP \equiv C .$$

Thus $C = D^+$ . Let $R = I - DD^+$ . Then by (103),

$$R = P - Pa(a^TPa)^{-1}a^TP$$

and

$$\text{Rank}(R) \geq \text{Rank } P - 1 .$$

Note that $Rx = 0$ for all $x$ satisfying $Px = 0$ . Now by (96) and (97), $P(Pu) = Pu \neq 0$ , but

$$R(Pu) = Pu - Pa(a^TPa)^{-1}(a^TPa)\ 2$$

$$= 0 \qquad \text{by (102)} ,$$

which implies

$$= AA^+ - AA^+u(2a^TPa)^{-1}a^TP$$

$$+ u\frac{1}{2k}u^TA^+ - u\frac{1}{2k}u^TPa(2a^TPa)^{-1}u^TA^+$$

$$- u\frac{1}{2k}u^TA^+u(2a^TPa)^{-1}a^TP$$

$$+ u\frac{1}{2k}u^TPa\left(k(a^TA^+b + 1)\right)(a^TPa)^{-2}a^TP$$

$$= AA^+ - AA^+u(2a^TPa)^{-1}a^TP$$

$$+ u\left(\frac{1}{2k} - \frac{1}{2k}\right)u^TA^+$$

$$+ u\left(\frac{1}{2k}\left(-u^TA^+u(2a^TPa)^{-1} + 4k(a^TA^+b + 1)(2a^TPa)^{-1}\right)\right)a^TP$$

Now $u^TA^+u = a^TA^+a + k^2b^TA^+b + 2k(a^TA^+b)$ , and by (99) it follows that

$$a^TA^+a + k^2b^TA^+b = 2k(a^TA^+b + 1) \Rightarrow u^TA^+u = 4ka^TA^+b + 2k$$

So

$$DC = AA^+ - AA^+u(2a^TPa)^{-1}a^TP$$

$$+ u\left(\frac{1}{2k}(2a^TPa)^{-1}\left(4k(a^TA^+b - a^TA^+b) + 2k\right)\right)a^TP$$

$$= AA^+ + (I - AA^+)a(a^TPa)^{-1}a^TP$$

$$= AA^+ + Pa(a^TPa)^{-1}a^TP \ , \tag{103}$$

which by (5) is symmetric. Note also that $C = C^T$ and $D = D^T$ . Then from (13) and (103) it follows that

$$DC = CD = (CD)^T$$

and (4) holds. Then

$$DCD = \left(A + u\frac{1}{2k}u^T\right)\left(AA^+ + Pa(a^TPa)^{-1}a^TP\right)$$

$$= AAA^+ + u\frac{1}{2k}u^TAA^+ + u\frac{1}{2k}u^TPa(a^TPa)^{-1}a^TP$$

then

$$\mathrm{Rank}(B) = \mathrm{Rank}(A) \tag{100}$$

and

$$\begin{aligned} B^+ = D^+ &- D^+D^+v(v^TD^+D^+v)^{-1}v^TD^+ \\ &- D^+v(v^TD^+D^+v)^{-1}v^TD^+D^+ \\ &+ D^+v(v^TD^+D^+v)^{-2}(v^TD^+D^+D^+v)v^TD^+ \\ = G \ , \end{aligned} \tag{101}$$

where

$$k = \frac{a^TPa}{a^TPb} = \frac{a^TPb}{a^TPb} \ ,$$

$$u = a + bk \ ,$$

$$v = a - bk \ ,$$

$$D = A + u\,\frac{1}{2k}\,u^T \ ,$$

and

$$\begin{aligned} D^+ = A^+ &- Pa(2a^TPa)^{-1}u^TA^+ \\ &- A^+u(2a^TPa)^{-1}a^TP \\ &+ Pa\big(k(a^TA^+b + 1)\big)(a^TPa)^{-2}a^TP = C \ . \end{aligned}$$

<u>Proof</u>: Note first the following:

(i) From (96) it follows that $a^TPa = a^TPPa > 0$ and so has an inverse.

(ii) From (98) it we get $u^TPa = 2a^TPa$ , and $Pv = 0$ .

$$u^TPa = 2a^TPa \quad \text{and} \quad Pv = 0 \ . \tag{102}$$

(iii) $DC = \left[A + u\,\frac{1}{2k}\,u^T\right]\Big(A^+ - Pa(2a^TPa)^{-1}u^TA^+ - A^+u(2a^TPa)^{-1}a^TP$

$+ Pa\big(k(a^TA^+b + 1)\big)(a^TPa)^{-2}a^TP\Big)$

$$+ Pu\,\frac{1}{2}\,\psi^{-2}\lambda^{-1}k^2\left(a^TA^+ab^TA^+b - (a^TA^+b + 1)^2\right)a^TP$$

$$= A^+ - A^+v\lambda^{-1}v^TA^+ - (A^+ - A^+v\lambda^{-1}v^TA^+)u\,\frac{1}{2}\,\psi^{-1}\lambda^{-1}a^TP$$

$$- Pa\,\frac{1}{2}\,\psi^{-1}\lambda^{-1}u^T(A^+ - A^+v\lambda^{-1}v^TA^+)$$

$$+ Pa\psi^{-2}\lambda^{-1}k^2\left(a^TA^+ab^TA^+b - (a^TA^+b + 1)^2\right)a^TP$$

$$\equiv G\ .$$

Let $Q = I - BG$ . Then by (95),

$$Q = P - Pu(u^TPu)^{-1}u^TP\ ,$$

and by (14)

$$\text{Rank}(Q) \geqslant \text{Rank}(P) - 1\ .$$

Now $P(Pu) = Pu \neq 0$ by (86) and (87), but

$$Q(Pu) = Pu - Pu(u^TPu)^{-1}u^TPu = 0\ .$$

$Qx = 0$ for all $x$ satisfying $Px = 0$ . Hence Nullity $(Q) \geqslant$ Nullity $(P) + 1$ , which implies

$$\text{Rank}(Q) = \text{Rank}(P) - 1$$

which gives

$$\text{Rank}(B) = \text{Rank}(A) + 1\ .$$

<u>Theorem 9</u>:

If

$$Pa \neq 0\ , \tag{96}$$

$$Pb \neq 0\ , \tag{97}$$

$$Pa = Pbk \quad \text{for some } k \neq 0\ , \tag{98}$$

and

$$(a-bk)^TA^+(a-bk) - 2k = 0 \tag{99}$$

Now $B = B^T$ , $G = G^T$ , $BG = (BG)^T$ , and (13) $\rightarrow GB = BG = (GB)^T$ , so (5) holds. So

$$BGB = [AA^+ + Pu(u^TPu)^{-1}u^TP][A + u\frac{1}{2k}u^T - v\frac{1}{2k}v^T]$$

$$= AA^+A + AA^+u\frac{1}{2k}u^T - v\frac{1}{2k}v^T + Pu(u^TPu)^{-1}(u^TPu)\frac{1}{2k}u^T$$

$$= A + (I - A^+A + A^+A)u\frac{1}{2k}u^T - v\frac{1}{2k}v^T$$

$$= A + u\frac{1}{2k}u^T - v\frac{1}{2k}v^T$$

$$\equiv B .$$

And

$$\begin{aligned} GBG = [A^+A + Pu(u^TPu)^{-1}u^TP][&A^+ - A^+v\lambda^{-1}v^TA^+ \\ &- Pa\psi^{-1}\lambda^{-1}\frac{1}{2}u^T(A^+ - A^+v\lambda^{-1}v^TA^+) \\ &- (A^+ - A^+v\lambda^{-1}v^TA^+)u\frac{1}{2}\lambda^{-1}\psi^{-1}a^TP \\ &+ Pa\psi^{-2}\lambda^{-1}k^2(a^TA^+ab^TA^+b - (a^TA^+b + 1)^2)a^TP] \end{aligned}$$

$$\begin{aligned} = {} & A^+AA^+ - A^+AA^+v\lambda^{-1}v^TA^+ \\ & + A^+A(A^+ - A^+v\lambda^{-1}v^TA^+)u\frac{1}{2}\lambda^{-1}\psi^{-1}a^TP \\ & - Pu(u^TPu)^{-2}u^TPu\lambda^{-1}u^T(A^+ - A^+v\lambda^{-1}v^TA^+) \\ & + Pu(u^TPu)^{-1}(u^TPu)\frac{1}{2}\psi^{-2}\lambda^{-1}k^2(a^TA^Aab^TA^+b - (a^TA^+b + 1)^2)a^TP \end{aligned}$$

$$\begin{aligned} = {} & A^+ - A^+v\lambda^{-1}v^TA^+ \\ & - (A^+ - A^+v\lambda^{-1}v^TA^+)u\frac{1}{2}\psi^{-1}\lambda^{-1}a^TP \\ & - Pu\frac{1}{4}\psi^{-1}\lambda^{-1}u^T(A^+ - A^+v\lambda^{-1}v^TA^+) \end{aligned}$$

$$(I - DD^+)v = 0 \quad \text{and}$$

$$v^TD^+v \neq 2k \ ,$$

and by the interlocking property of eigenvalues $B$ will be indefinite if

$$v^TD^+v > 2k \ . \tag{109}$$

Now by (108)

$$v^TD^+v = v^TA^+v + 2u^TA^+v + u^TA^+u + 2k$$

$$= 4a^TA^+a + 2k > 2k \ , \quad \forall\ k$$

by Lemma 1. Hence (109) holds.

Corollary 3:

If

$$Pa \neq 0 \ ,$$

and

$$Pb \neq 0 \ ,$$

then necessary and sufficient conditions for $B$ to be positive semidefinite are:

(i) $Pa = Pbk$ for some $k > 0$ , and

(ii) $(a - bk)^TA^+(a - bk) \leqslant 2k$ (110)

Proof: By contraposition. There are three cases.

(i) Suppose $Pa \neq Pbk$ for all $k$ . Then the conditions of Theorem 7 hold. Define

$$u = a + bk \quad \text{and}$$

$$v = a - bk \quad \text{for some} \quad k > 0 \ .$$

Then $Pu \neq 0$ , $u \frac{1}{2k} u^T$ is a positive dyad, $D = A + u \frac{1}{2k} u^T$

will be positive semidefinite, and $\mathrm{Rank}(D) = \mathrm{Rank}(A) + 1$ . Now by (82) $\mathrm{Rank}(B) = \mathrm{Rank}(A) + 2$ , which will occur if and only if

$$(I - D^+D)v \neq 0 .$$

But $v\left(-\frac{1}{2k}\right)v^T$ is a negative dyad. This with the above gives $Pa \neq Pbk$ for all $k$ if and only if B is indefinite.

(ii) Suppose $Pa = Pbk$ for some $k < 0$ . Then the conditions of Theorem 8 hold. Define $u$ , $v$ , and $k$ as in Theorem 8. Then we have $v\left(-\frac{1}{2k}\right)v^T$ is a positive dyad, $Pv = 0$ , $v^TA^+v - 2k > 0$ , and by Theorem 10, $D = A - v\frac{1}{2k}v^T$ remains positive semidefinite, and $\mathrm{Rank}(D) = \mathrm{Rank}(A)$. By (89) $\mathrm{Rank}(B) = \mathrm{Rank}(A) + 1$ , which occurs if and only if $(I - DD^+)u \neq 0$ . But for $k < 0$ , $u\frac{1}{2k}u^T$ is a negative dyad. This with the above gives for $Pa \neq 0$ , $Pb \neq 0$ , $Pa = Pbk$ for $k < 0$ if and only if B is indefinite.

(iii) Suppose $Pa = Pbk$ for some $k > 0$ and (110) does not hold. Then the conditions of Theorem 8 hold. Define $u$ , $v$ and $k$ as in Theorem 8. Then $Pu \neq 0$ , $Pv = 0$ , $D = A + u\frac{1}{2k}u^T$ will be positive semidefinite, and $\mathrm{Rank}(D) = \mathrm{Rank}(A) + 1$ . Now (89) gives $\mathrm{Rank}(B) = \mathrm{Rank}(A) + 1$ , which occurs if and only if

$$(I - D^+D)v = 0 .$$

$B = D - v\frac{1}{2k}v^T$ will be indefinite if and only if (110) does not hold. Hence for $Pa \neq 0$ , $Pb \neq 0$ ,

$Pa = Pbk$ for $k > 0$ and $v^T A^+ v > 2k$

if and only if $B$ is indefinite. This concludes the proof.

It will also be useful to note that eigenvectors $(e_1, e_2)$ and associated eigenvalues $(\lambda_1, \lambda_2)$ for $ab^T + ba^T$ are

$$e_1 = \frac{a}{\|a\|} + \frac{b}{\|b\|}, \qquad \lambda_1 = a^T b + \|a\|\|b\| \quad \text{and}$$

$$e_2 = \frac{a}{\|a\|} - \frac{b}{\|b\|}, \qquad \lambda_2 = a^T b - \|a\|\|b\|$$

<u>Proof</u>:

$$(ab^T + ba^T)e_1 = \frac{a}{\|a\|} + \frac{b}{\|b\|}(a^T b + \|a\|\|b\|) = e_1\lambda_1$$

$$(ab^T + ba^T)e_2 = \frac{a}{\|a\|} - \frac{b}{\|b\|}(a^T b - \|a\|\|b\|) = e_2\lambda_2 .$$

Note that $ab^T + ba^T$ can be expressed as the sum of two symmetric dyads involving $e_1$ and $e_2$, namely,

$$ab^T + ba^T = e_1 \frac{\|a\|\|b\|}{2} e_1^T + e_2\left(- \frac{\|a\|\|b\|}{2}\right)e_2^T ,$$

and that the first of these is the positive part of $(ab^T + ba^T)$ .

An algorithm for estimating the positive part of a symmetric matrix $A$ follows from the preceding results.

We assume that $A$ is given as

$$A = \sum_{i=1}^{Q} d_i c_i d_i^T + \sum_{j=1}^{R} (a_j b_j^T + b_j a_j^T)$$

$$+ \sum_{\ell=1}^{S} t_\ell r_\ell t_\ell^T ,$$

where

$$d_i\ , \quad i = 1,\ldots,Q\ ,$$

$$a_j, b_j\ , \ j = 1,\ldots,R\ , \text{ and}$$

$$t_\ell\ , \quad \ell = 1,\ldots,S \quad \text{are} \quad n \times 1 \quad \text{vectors,}$$

and

$$c_i > 0\ , \ i = 1,\ldots,Q \quad \text{and}$$

$$r_\ell < 0\ , \ \ell = 1,\ldots,S \quad \text{are scalars.}$$

Denote

$$A_k = \sum_{i=1}^{\min(k,Q)} d_i c_i d_i^T + \sum_{j=1}^{\min(k-Q,R)} (a_j b_j^T + b_j a_j^T) \cdot I(k > Q)$$

$$+ \sum_{\ell=1}^{k-Q-R} t_\ell r_\ell t_\ell^T \cdot I(k > Q + R)\ ,$$

where $I(\cdot)$ is the indicator function,

$$I(s) = \begin{cases} 1 \text{ if } s \text{ is true} \\ 0 \text{ if } s \text{ is false,} \end{cases}$$

and denote $P_k = I - \mathcal{A}_k \mathcal{A}_k^+$ .

With each iteration the algorithm forms an estimate $\mathcal{A}_k$ of the positive part of $A_k$ and updates its generalized inverse. If during the course of processing rank two perturbations, a summand is encountered which cannot be entirely absorbed without yielding $\mathcal{A}_k$ indefinite or dropping the rank of $\mathcal{A}_k$ , the summand is separated into two symmetric parts (one positive dyad and one negative dyad), and the positive dyad is absorbed. The negative dyad is then added to the set of negative dyads for possible absorption after all rank two summands have been processed. The algorithm proceeds in the following manner:

•A• Initialization

Set $k = 1$ , $A_0 = 0$ , $M = Q + R + S$ , $Y = \phi$ , $Z = \phi$ .

•B• Iteration k:

If $k > Q$ go to •C•.

Set $A_k = A_{k-1} + d_k c_k d_k^T$ .

Update $A_k^+$ . Set $k = k + 1$ and go to •B•.

•C• If $k > M$ go to •H•.

If $k > Q + R$ go to •F•.

Set $\ell = k - Q$ .

If $P_{k-1}\, a_\ell \neq 0$ go to •D•.

If $P_{k-1}\, b_\ell \neq 0$ go to •E•.

Compute $\alpha = a_\ell^T A_{k-1}^+ a_\ell \cdot b_\ell^T A_{k-1}^+ b_\ell - (a_\ell^T A_{k-1}^+ b_\ell + 1)^2$ .

If $\alpha \geqslant 0$ go to •E•.

Set $A_k = A_{k-1} + a_\ell b_\ell^T + b_\ell a_\ell^T$ .

Update $A_k^+$ , set $k = k + 1$ and go to •C•.

•D• If $P_{k-1} b_\ell = 0$ go to •E•.

Compute $\beta = a_\ell^T P_{k-1} b_\ell / b_\ell^T P_{k-1} b_\ell$ .

If $\beta < 0$ go to •E•.

If $P_{k-1} a_\ell \neq P_{k-1} b_\ell \cdot \beta$ go to •E•.

Compute $\gamma = (a_\ell - b_\ell \cdot \beta)^T A_{k-1}^+ (a_\ell - b_\ell \cdot \beta) - 2\beta$ .

If $\gamma > 0$ go to •E•.

Set $A_k = A_{k-1} + a_\ell b_\ell^T + b_\ell a_\ell^T$ .

Update $A_k^+$ , set $k = k + 1$ and go to •C•.

•E• Compute

$$u = a \cdot \frac{1}{\|a\|} + b \cdot \frac{1}{\|b\|} ,$$

$$v = a \cdot \frac{1}{\|a\|} - b \cdot \frac{1}{\|b\|} \text{ , and}$$

$$\pi = \frac{\|a\|\|b\|}{2} .$$

Set $\mathcal{A}_k = \mathcal{A}_{k-1} + u\pi u^T$ .

Update $\mathcal{A}_k^+$ , set $k = k + 1$ .

Set $M = M + 1$ , $t_M = v$ , $r_M = -\pi$ , and go to •C•.

•F• If $k > M$ go to •H•.

Set $\ell = k - Q - R$ .

If $P_{k-1}t_\ell = 0$ go to •G•.

Set $\mathcal{A}_k = \mathcal{A}_{k-1}$ , $\mathcal{A}_k^+ = \mathcal{A}_{k-1}^+$ , $Y = Y \cup \{\ell\}$ .

Set $y_\ell = P_{k-1}t_\ell$ , $k = k + 1$ and go to •F•.

•G• Compute $\hat{r}_\ell = \max\left(r_\ell, -(t_\ell^T \mathcal{A}_{k-1}^+ t_\ell)^{-1}\right)$ .

If $\hat{r}_\ell = -(t_\ell^T \mathcal{A}_{k-1}^+ t_\ell)^{-1}$ , set $Z = Z \cup \{\ell\}$ , $z_\ell = \mathcal{A}_{k-1}^+ t_\ell$ .

Set $\mathcal{A}_k = \mathcal{A}_{k-1} + t_\ell \hat{r}_\ell t_\ell^T$ .

Update $\mathcal{A}_k^+$ , set $k = k + 1$ , and go to •F•.

•H• Set $\mathcal{A} = \mathcal{A}_M$ . End.

In the manner of Emami and McCormick (1978) and Sofer (1984), we can observe the following:

Note that for $k > Q + R$ ,

$$A = \mathcal{A}_{k-1} + \sum_{j=k}^{M} t_j r_j t_j^T .$$

Since $r_j < 0$ for all $j > Q + R$ , we see that

$$x^T A x \leqslant x^T \mathcal{A} x \leqslant x^T \mathcal{A}_k x \text{ , } \forall \; x \text{ , } k > Q + R . \tag{111}$$

Thus $\mathcal{A}$ may be considered an overestimate of $A$ in that $\mathcal{A} - A$ is

positive semidefinite, and we find that all directions of positive curvature for $A$ are also directions of positive curvature for $\bar{A}$ and all directions of zero curvature for $\bar{A}$ are directions of nonpositive curvature for $A$ .

At the conclusion of the algorithm the set $Y$ will contain the indices of those negative dyads (some of which were separated out from rank two summands) which could not be absorbed, and the set $Z$ will contain the indices of those negative dyads which caused a rank decrease in forming $\bar{A}$ .

Now for each $y_\ell = P_{k-1}t_\ell$ , $\ell \in Y$ and $k = Q + R + \ell$ , it follows from (111) that

$$y_\ell^T A y_\ell \leq y_\ell^T A_{k-1} y_\ell + (y_\ell^T t_\ell) r_\ell (t_\ell^T y_\ell) = (t_\ell^T P_{k-1} t_\ell)^2 r_\ell < 0 .$$

Hence each $y_\ell$ will be a direction of negative curvature for $A$ .

Also note that by construction, the vectors $z_\ell$ , $\ell \in Z$ , are directions of zero curvature for $A_k$ , $(k = Q + R + \ell)$ , since for each $\ell$

$$z_\ell^T A_k z_\ell = t_\ell^T A_{k-1}^+ \left[ A_{k-1} + t_\ell \left( -(t_\ell^T A_{k-1}^+ t_\ell)^{-1} t_\ell^T \right) \right] A_{k-1}^+ t_\ell = 0 . \quad (112)$$

So from (112) we see that

$$z_\ell^T A z_\ell \leq z_\ell^T A_k z_\ell + z_\ell^T t_\ell (r_\ell - \hat{r}_\ell) t_\ell^T z_\ell$$

$$= (t_\ell^T A_{k-1}^+ t_\ell)^2 (r_\ell - \hat{r}_\ell) \leq 0 , \quad \ell \in Z ,$$

and each $z_\ell$ will be a direction of nonpositive curvature for $A$ . If $r_\ell < \hat{r}_\ell$ , it will be a direction of negative curvature for $A$ .

From (111) we also see that any direction of zero curvature for $A_k (k > Q + R)$ is also a direction of zero curvature for $\bar{A}$ and so all directions $x \in \{z_\ell, \ell \in Z\} \cup \{y_\ell, \ell \in Y\}$ are directions of zero curvature for $\bar{A}$ .

The preceding algorithm is much in the spirit of that of Emami and McCormick (1978) with the following exceptions:

(a) The Emami-McCormick algorithm presupposes that all rank two summands have been decomposed in some manner into symmetric dyads. In the algorithm above, rank two matrices are decomposed and the positive part absorbed only if absorbing the whole summand at that stage of the algorithm would decrease the rank of $A$ or leave it indefinite.

(b) Rank two summands, when decomposed, are divided into positive and negative parts employing their spectral decomposition. Thus at any stage of the algorithm, if only part of a rank two summand can be absorbed, "all" of its positive part is absorbed. It is hoped that this approach will yield in some sense a more accurate estimate of the positive part of $A$ . That it will in the trivial case can be seen by the following example.

Suppose $A = 0$ , $a = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Then $A + ab^T + ba^T$ is indefinite, with one positive and one negative eigenvalue.

Now also suppose that $(ab^T + ba^T)$ is decomposed into two symmetric dyads

$$u \frac{1}{2} u^T + v\left(- \frac{1}{2}\right)v^T \text{ , where}$$

$$u = a + b = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \text{ and } v = a - b = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

(this is commonly done). By the algorithm above,

At Iteration 1,

$P_0 = I$ , $P_0 a = a \neq 0$ , $P_0 b = b \neq 0$ ,

$P_0 a \neq P_0 bk$ for all $k$ ,

$\|a\| = 5$ , $\|b\| = 1$ .

So

$$A_1 = \left(\begin{pmatrix} .6 \\ .8 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) 2.5 \left(\begin{pmatrix} .6 \\ .8 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)^T$$

$$= \begin{pmatrix} .6 \\ .8 \end{pmatrix} 2.5\ (.6, 1.8) .$$

$$A_1^+ = \begin{pmatrix} .6 \\ 1.8 \end{pmatrix} .0309\ (.6, 1.8) .$$

$$M = M + 1 = 1 , \quad t_1 = \begin{pmatrix} .6 \\ -.2 \end{pmatrix} , \quad r_1 = -2.5 .$$

At Iteration 2,

$$P_1 = (I - A_1 A_1^+) = \begin{bmatrix} .9 & -.3 \\ -.3 & .1 \end{bmatrix}$$

$$P_1\ t_1 = \begin{pmatrix} .6 \\ -.2 \end{pmatrix} \neq 0$$

and the algorithm terminates with $A = A_1$ = the positive part of $A$ .

By the Emami-McCormick algorithm,

At Iteration 1,

$P_0 = I$ , $Pu = u \neq 0$ , so

$$A_1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \frac{1}{2}\ (3,5) \text{ , and}$$

$$A_1^+ = \begin{pmatrix} 3 \\ 5 \end{pmatrix} .0017\ (3,5) .$$

At Iteration 2,

$$P_1 = \begin{bmatrix} .14706 & -.44118 \\ -.44118 & .26471 \end{bmatrix} , \quad P_1 v = \begin{pmatrix} -.44118 \\ -.52941 \end{pmatrix} \neq 0 .$$

The negative dyad cannot be absorbed and the algorithm terminates with the estimate $A = A_1 \neq$ the positive part of $A$ .

## References

Ben-Israel, A. and T. N. E. Greville (1974). *Generalized Inverses: Theory and Applications.* Wiley, New York.

Emami, G. and G. P. McCormick (1978). Use of a stable generalized inverse algorithm to evaluate Newton method strategies. Technical Paper Serial GWU/IMSE/T-384/78, Institute for Management Science and Engineering, George Washington University.

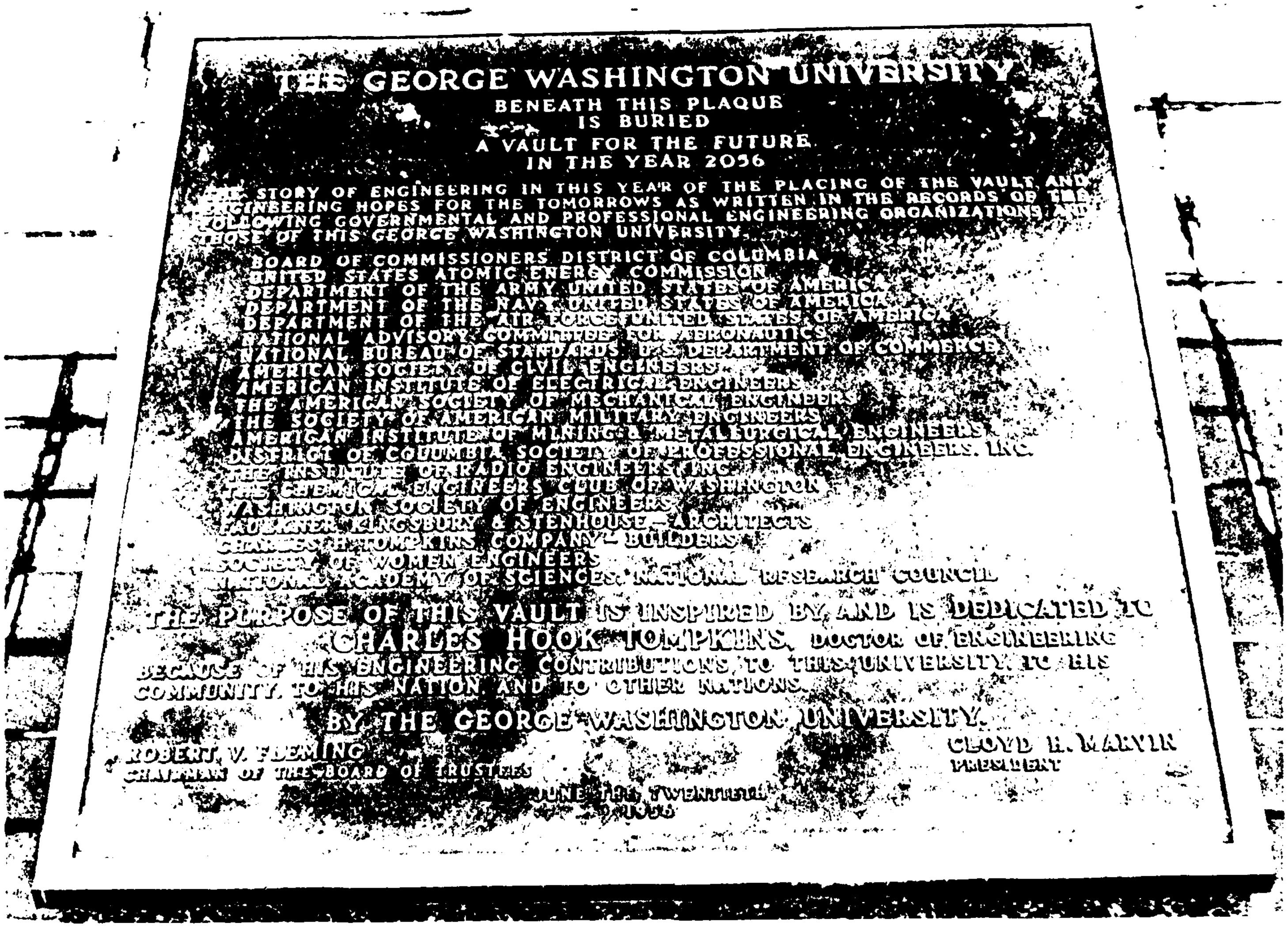
Halmos, P. (1958). *Finite Dimensional Vector Spaces.* Van Nostrand, Princeton.

Leaver, S. G. (1984). Computing global maximum likelihood parameter estimates for product models for frequency tables involving indirect observation. D.Sc. Dissertation, George Washington University.

McCormick, G. P. (1976). Formulas for updating the generalized inverse of a matrix perturbed by a dyad. Technical Paper Serial GWU/IMSE/T-333/76. Institute for Management Science and Engineering, George Washington University.

Meyer, C. D., Jr. (1973). Generalized inversion of modified matrices. *SIAM J. Appl. Math.*, *24*, 315-323.

Sofer, A. (1984). Efficient matrix methods for solving nonlinearly constrained optimization problems via Newton's method when the projected Lagrangian Hessian is given in dyadic form. D.Sc. Dissertation, George Washington University.

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